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THEORY OF VISCOUS-FLUID JETS

by

L. A. Vulis and V. P. Kashkarov

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L. A. Vulis i V. P. Kashkarov

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ABSTRACT: This book will be of particular interest to persons concerned with the problems of fluid jet streams. The book is devoted to the results of investigations of a broad and widespread category of incompressible fluid motions in the form of laminar and turbulent jets. The development of computational methods applicable to an important type of jet streams and based on a consistent and systematic study of jet flows with a theoretical approach is the aim of this monograph. There are four parts to the book, including a foreword and an introduction. The first part deals with the solutions of jet problems based on the exact Navier-Stokes equations for incompressible fluids and, in particular, with the Landau investigation of the propagation of a submerged axially symmetric viscous fluid jet issuing from a thin tube. The second part contains a detailed analysis of laminar jet streams of an incompressible fluid by methods of boundary layer theory. In addition to free jets flowing into a stationary medium or into homogeneous wake flows, semibounded jets (wall jets) are considered. Turbulent jets of liquids and gases are the subjects of the third part, in which self-similar solutions for free and wall jet sources are investigated. It also contains detailed experimental data obtained under the guidance of one of the authors in thermophysical laboratories in Alma Ata for comparison with the theoretical results. The fourth part deals with certain theoretical and experimental problems of jet streams which may be regarded as complementary to the main topics treated in this book. Among them are complex turbulent jet streams, patterns of diffusion flames, and jets in magnetohydrodynamics. The authors thank K. E. Dzhaugashtin and L. P. Yarin for their help in selecting data and in writing Chapters 17 and 18, and also G. N. Abramovich and G. Yu. Stepanov for their comments.

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PREFACE

This monograph calling for the reader's attention has been devoted to the results of investigations of a broad and widespread class of motions of a viscous incompressible liquid or a compressible gas, namely the laminar and turbulent flows. All considerations apply to steady, unstressed motions of a continuous medium.

The interest aroused by jet flows is explained by the great importance of these flows for so many fields of technical engineering. In rockets, airplanes and engines, in turbines and boilers, combustion chambers, burners and furnaces, in hydraulic works, chemical and technological devices, ventilators, devices of jet automation (pneumonics), etc., we are concerned with jet flows of liquids or gases. They are, as a rule, essential and often have a decisive importance for the intensification of the operational process or its efficiency.

Apart from this, the jet motions take a conspicuous place in the theoretical and, particularly, in the applied mechanics of viscous fluids. The relevant literature is extensive and comprehensive. In the last two or three decades the number of publications dealing with the problems of jet motions has increased rapidly in both the Soviet Union and abroad. Among the publications on jets there are special monographs on the theory and the results of experiments, review articles in manuscripts on hydro- and gas-dynamics, in all a large number of articles. The references to this literature have been compiled at the end of the present book.

The great variety in the ways of treatment of the jet motions is

characteristic of the investigations carried out by the various authors. Some of them consider the problem from a purely theoretical point of view, others on the basis of a purely experimental conception, while in many other papers both aspects have been fused to some degree. In some papers the authors have restricted themselves to investigating turbulent jets which are of immediate practical importance, in others they have considered mainly laminar jets which are based on the strict equations of hydrodynamics.

In this connection we must briefly summarize the method of investigating the jet motions chosen in the present book. The principal goal of the book has been a consequent and systematic consideration from an - as far as possible - unique point of view of the theory on jet motions and, based on this theory, of the main development of mathematical methods applying to the types of flows which are of practical importance. It is, of course, the turbulent jets which are mainly important in practice. Just as in other fields of the mechanics of viscous fluids, the theoretical model of turbulent jets may be built up in the usual manner from the laws of expansion of laminar jets. Their consideration which is also interesting in itself will, expediently, begin with an analysis of the solutions obtained on the basis of the exact equations of the hydrodynamics of a viscous fluid, the Navier-Stokes equations. This enables us to ascertain a series of important general properties of jet motions which are essential not only for an investigation of the laminar jet flows by the methods of the boundary layer theory, but also for problems on turbulent jet flows. Thus, whenever this is possible, a solution of the latter will be based on the qualitative flow pattern which is, first of all, obtained from an analysis of the corresponding laminar motion.

At the same time the results of a direct experiment have a deci-

sive importance for the investigation of turbulent flows. They serve as a starting point for the development of a mathematical procedure for the concrete motion and as a criterion for the usefulness of this procedure in the final state of the investigations. The methods of calculation which are applied at present in the theory of turbulent flows and which are based on the so-called semiempirical theories, are not universal. This indicates that the solution of some problems can be obtained with about the same accuracy on the basis of different assumptions on the mechanism of turbulent transport while a large group of them are solved simply with the help of various mathematical methods. At the same time it very often happens that for a related problem none of these methods is sufficiently effective. The empirical constants derived from the experiment are likewise nonuniversal; they characterize the influence of various factors which are not taken into account explicitly (in particular the microstructure of turbulent flow). Unfortunately, the statistical branch in the theory of turbulence, which is most promising in the general plan, has so far yielded no results which are of practical importance for the theory and the calculation of jets. The application of the statistical theory of turbulence to jet motions has therefore not been discussed in the present book.

The necessity of solving various problems encountered in engineering applications forces us in the present book to cover not only the well-known and, maybe, classical methods of the theory of the turbulent boundary layer. In addition there are various semiempirical and purely empir cal mathematical procedures or even the mere results of an experiment which deserve our interest. This causes a considerable extension of the range of problems which can be solved by a preliminary approximation method and yields additional material for subsequent generalizations. In the complex and far from complete process of

investigating jet flows not a single extreme case can be considered justified. Neither the abandonment of the necessary practical problems for the "clean" region of the analytical solutions, nor the rejection of generalizations of the laws derived from the individual experiments carried out under concrete conditions, enables us to develop a sufficiently broad theory meeting the requirements of practice.

There general considerations and also the choice of the material for the book and its entire structure characterize the scientific direction of the authors. In the treatment of the problem the authors tended to a more complete analysis of the physical nature of the phenomena, to a simplified application of the material and, in this connection, to its representation in the form of mathematical results and illustrations. The solutions of various problems have been compiled in detailed tables for the same purpose. To restrict the volume of the book some problems which are analyzed in detail in literature accessible for a large circle of readers have been considered in a condensed form and sometimes even omitted. The results obtained by the authors themselves have been considered in greater detail than elsewhere. The sections dealing with turbulent flows also contain experimental data in sufficient detail for comparison with the results of calculations. Most of the data have been taken from papers produced under the guidance of one of the authors between 1951 and 1963 in the thermophysical laboratories of Alma-Ata.

Let us now briefly discuss the material in this book; for details we refer to the Table of Contents.

The first part is devoted to a detailed discussion of the solutions to flow problems based on the exact Navier-Stokes equations for an incompressible fluid. In this part the well-known investiga-

tions by L.D. Landau are considered as fundamental; they deal with the laws governing the expansion of an axisymmetric jet of viscous fluid discharged from a thin tube into another fluid. The flow pattern in the jet and the heat (or mass) transfer are discussed in detail, indicating the fundamental characteristic features of the motion. The dynamic and thermal problems (under different boundary conditions) on the expansion of fan-type jets are also considered. For the so-called "intense" jet the transition was made to a solution within the framework of the theory of the laminar boundary layer.

The second part of the contains a detailed analysis of the methods of the boundary layer theory of laminar jet flows of an incompressible liquid and a gas. Apart from the free jets discharged into an immobile medium or a uniform comoving flow, we consider the "half-limited" jets (expanding along a solid wall). Both the dynamic and the thermal problems are solved with the help of the method of the asymptotic boundary layer which is used throughout the book. Particular attention is paid to compressible-gas jets.

The third part is devoted to turbulent flows of liquids and gases. We consider in detail the self-similar solutions for both free and half-limited source jets of an incompressible liquid. On the basis of the hypothesis on the similarity of the momentum flux density distributions a generalization is given of the data on the free compressible-gass jets. Great attention is paid to the method of the equivalent heat-conduction problem which permits a detailed investigation into the problem of the expansion of liquid or gas jets discharged from nozzles of finite dimensions with an arbitrarily chosen initial velocity distribution, temperature distribution, etc. The results of the calculations are compared with the experimental

ones.

The fourth part deals with some theoretical and experimental problems in the field of jet flows which, for some reason or other are supplementary to the basic material of the book. Here we also give a brief review on some complex turbulent jet flows (counterflows and jet flows around bodies). The theory of such motions has hardly been developed as yet. Chapter 16 will therefore mainly contain experimental data and attempts of their primary generalization which are valuable in connection with the application of such flows. The next chapter, 17, in contrast to the previous chapter, gives an example of a successful application of the theory of laminar and turbulent jets to a special problem: the calculation of a diffusion gas flare. That it was entered in the fourth part of the book is due to the fact that a more complete representation of the flare theory (homogeneous, indirect-jet flare, etc.) and also of problems specific for the combustion theory are beyond the scope of this book. Finally, the last chapter contains a few magnetohydrodynamic jet problems which are mainly interesting insofar as they can be treated by the same methods as ordinary jets of a non-conducting viscous fluid.

The list of references on viscous-fluid jets given at the end of the book is incomplete. It comprises first of all investigations cited in the text (in particular, a detailed leterature list of the papers by the heat physicists of Alma-Ata has been given, papers which have mainly appeared in the limited circulation of the literature of the Republic). In addition, the reference list also contains review review articles and jet flow investigations known to the authors, the acquaintance of which is essential for the completeness of a representation of the problem's present state. The papers published up to 1963 are listed more completely as the manuscript of the book was then

finished in its basic form. At the end of each chapter literature references are given which refer to this chapter in particular.

For convenience we have chosen for the book a continuous numeration of the chapters and a dual for the sections, figures, tables and formulas (the First Digit indicates the chapter, the second, e.g., the formula within this chapter).

Not all the problems treated in the book may be considered as having reached the same degree of solution. Some of them need a theoretical or experimental verification. The main direction must be a widening of the circle of important practical problems which can be treated mathematically by the methods of the theory of viscous-fluid jets, the development of these methods, an ascertainment of their limits of applicability, and the like.

The authors thank their colleagues for helping them with selection of the material, in particular K.Ye. Dzhaugashtin and L.P. Yarin who took part in the compilation of Chapters 18 and 17, respectively.

With deep gratitude the authors acknowledge the interesting discussions with G.N. Abramovich on a series of problems considered in the book. The authors thank G.N. Abramovich and G.Yu.Stepanov for their comments on the manuscript. The attentive work by V.Z. Parton helped to reduce the editing errors to a minimum.

All remarks which are intended to eliminate failings fo the book will accepted gratefully by the authors.

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V. Kashkarov

INTRODUCTION

0.1. THE SOURCE JET

The jet motions of a viscous liquid or gas may be of a very different nature as regards the form of flow, the kind of the fluid in the jet, the surrounding medium or other features. A brief systematic compilation of these motions referring to the contents of the present book will be given in the second section of the Introductionl It may be preceded by some information of general character.

Just as in any other theory, the theory of jets has its elementary models and techniques. Prior knowledge of these facilitates the investigation of more complex flows but is also interesting in itself. For an analysis of the jet motions of a viscous fluid it is in particular the idea of a unique source jet which is important. Many analytical solutions of the problems of the expansion of a jet refer, in fact, to motions caused by the source jet. This is the case, e.g., with a considerable part of the so-called "self-similar" solutions (sometimes referred to the literature as "similar" solutions).

The motion produced by a source jet closely resembles a real flow in a jet at a great distance from the nozzle from which it issues. Another valuable jet-flow model may be the flow in the mixing zone of a semi-infinite plane-parallel flow in a surrounding medium. Similar to this will be the motion in a plane or axisymmetric jet near the nozzle.* Before considering the peculiarities of the fluid's motion in the source jet often discussed in the following we turn to the qualitative

picture of the jet in a viscous fluid. We consider the latter by way of the particular example of a steady plane or axisymmetric fluid jet discharged into a unlimited medium at rest. For a medium with the same physical properties as those of the fluid in the jet, the expansion of the jet can be essentially reduced to a gradual leveling of the initial velocity distribution as shown schematically in Fig. 0.1. Owing to the (molecular and mainly turbulent) viscosity the jet draws the surrounding fluid into the motion and transfers part of its initial momentum to it. In this process the velocity at the jet axis and of course also in the cross sections will drop.

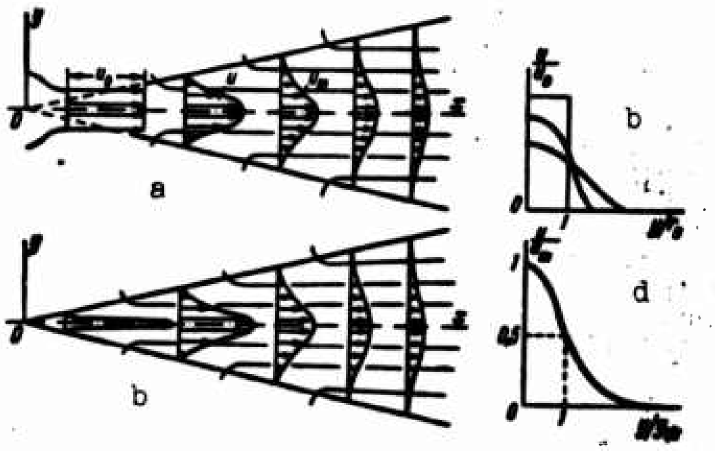

Fig. 0.1 Schematic representation of free jets. a)Jet of finite dimension, streamlines and velocity distributions in several cross sections of the jet; b)source jet; c)relative velocity distribution in a jet of finite dimensions (scale units: for the velocity the efflux velocity u_0; for the transverse coordinate the radius r_0 of the nozzle; d)relative velocity distribution in a source jet (scale units: for the velocity u_m on the axis of the jet; for the coordinates the value of $y_{\frac{1}{2}}$ at which $u = \frac{1}{2}u_m$).

A strict solution to the problem of the expansion of a jet would, obviously, require an integration of the equation of motion and the continuity equation (for simplicity we consider an incompressible fluid) with a given law of internal friction (molecular or molar turbu-

lent viscosity) and given boundary conditions. The latter must comprise the efflux conditions (the shape of the nozzle orifice and the velocity vector field at the outlet), and also the conditions in the zones of the nonperturbed fluid (theoretically at infinity) and, finally, the symmetry conditions for flows with plane or axial symmetry.

With such a general statement an analytical solution of the problem is as a rule connected with considerable difficulties, though in a series of practically important cases (e.g., the burning of a torch) this solution is of greatest interest.

Experimental observations and also theoretical considerations show that the laws governing a flow at a great distance from the nozzle display in some way a universal character. In this region the flow is virtually independent of the efflux conditions (in what follows we shall call them the "initial conditions"). This flow, at a great distance from the nozzle, where we may abstract from the concrete initial conditions, can be considered as the result of the action of a momentum point source (in the case of an axisymmetric jet, or a linear source for a plane jet) oriented in the direction of the axis of symmetry. Such a motion we shall call a source jet.

For an analytical solution of the source-jet problem the detailed initial conditions are replaced by a sufficiently integral condition whose part is usually played by a given characteristic quantity, the initial value of the total momentum flux J_x (projected onto the O_x symmetry axis):

$$J_x = \int\limits_{(s)} \rho u_{x_0}^2 ds,$$

where ρ is the density of the fluid, u_{x0} is the axial component of the velocity vector in the efflux section s of the true jet which can be replaced by the effective source.

It is easy to see that the concept of the source jet has a certain independent meaning in the jet theory; it may be considered without knowing the concrete form of the nozzle and the initial velocity distribution. It is in some way analogous to the concept of the mass point in mechanics (where at distances exceeding essentially the body dimensions, shape and dimensions of the body do not influence the law of motion, the field of gravity genereated by the body, etc.).

A characteristic example well-known from experiments is the fact that a jet flowing out of a quadratic, triangular or other form of orifice behaves at a sufficient distance from the nozzle virtually just like a jet ejected from a round orifice.

The practical significance of the concept of the source jet sonsists in the fact that the motion produced by it is self-similar. In the mathematical description this means a transition from partial differential equations to ordinary differential equations and with regard to the experiment it offers the possibility of generalizing the experimental results since the velocity profiles are similar (Fig. 0.1,b).

We must, however, stress the differences in the flows of a source jet of a viscous fluid and an ideal fluid. In the first case (source jet) the flow is characterized by its vector character, the directivity of the initial momentum which gives the whole motion an oriented character, a peculiar anisotropic flow. In contrast to this, the source usually considered in the theory of an ideal fluid produces an isotropic flow pattern.

It is extremely important that the initial momentum flux of a jet of viscous fluid, which expands in an unlimited nonmoving medium in the absence of external forces, is conserved as to magnitude and direction. This follows directly from the general momentum-conservation law.

The directivity of the flow produced by a source jet is closely

related with another characteristic property of the motion, namely the limitedness of the field of perturbations in the directions perpendicular to the initial momentum. As we see from Fig. 0.1,a the region of velocity change in a cross section of the jet (from maximum velocity to virtually zero at the nominal boundary of the jet) is relatively narrow; the transverse velocity gradients are much smaller than the longitudinal ones. The velocity components behave inversely: as a rule the transverse velocity components are considerably smaller than the longitudinal ones. These general properties of jet motions render it, fortunately, possible to apply the methods of the boundary layer theory.

In the case where the source jet together with the momentum flux introduces a flux of a certain property (an excessive heat content, concentration or the like) the process of expansion of the jet is accompanied by a dispersion of this property in the surrounding medium. In this dispersion the main part is played by the primary process, the momentum scattering. For the distributions of concentration, temperature, etc. the self-similarity of the flow, the characteristic properties of the boundary layer, etc. are also conserved under the corresponding boundary conditions. For an incompressible fluid the solution of the thermal problem (the problem of the temperature distribution) is built up on the basis of a preliminary solution to the dynamic problem (velocity distribution). In the case of a compressible gas both problems must be solved at the same time. In both cases, i.e., for the liquid and the gas, the ratio between the coefficients of momentum and heat (or mass) transfer, determined by the so-called Prandtl number is of decisive importance.

Here we briefly discussed the qualitative flow pattern by way of the example of a "submerged"* axisymmetric source jet. Our remarks ap-

ply, however, more or less to other more complex forms of jet motions of a viscous fluid. For these cases it is of course, necessary also to modify the geometrical from of the source jet. Besides the point source which produces an axisymmetric jet, we may be concerned with a plane source jet, a radial (or fan-type jet, i.e., a jet coming out of the gap between two discs), an annular jet, flowing out of the gap between two coaxial cylinders or cones and the like. One of the examples, the fan-type jet, is shown in Fig. 0.2. Just as in the case of the point source, the flows produced by sources of various forms will be characterized by integral characteristics, the momentum flux, the flux of heat content, etc.

Before we turn to a detailed analysis of concrete problems we think it expedient to give a brief systematic representation of the latter.

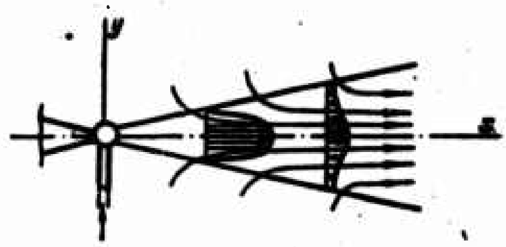

Fig. 0.2. Diagram of fan jet.

0.2. JET FLOWS

Let us try to classify as far as possible the various jet flows. The features according to which the various forms of jets must be distinguished are numerous. We shall only mention the most essential ones.

The plan of the present book has been based on the differences in the mechanism of jet expansion, that is, the mechanism of momentum, energy and mass transfer. In this respect it is, as always, ne~essary to distinguish between laminar and turbulent jets. In the first case the nature of the transfer effects such as molecular friction, thermal conductivity, diffusion, are well known. As to the second case, we do not

possess as yet any physically exact and closed system of equation to describe turbulent motion. In particular, it is for the present hardly possible to investigate systematically turbulent jets of fluid or gas on the basis of the modern statistical theory of turbulence. Sporadic attempts of applying statistical conceptions to the theory and calculation of turbulent jet (e.g., [159] and others) are, essentially, reduced to the same empirical methods. As already mentioned in the Preface, we shall not consider this direction in the present book.

From the general class of jet flows we may separate the motion of incompressible fluids which, as usual, comprises not only jets of droppable liquids but also jets of gases with relatively small density variations. The latter are encountered in the case of velocities which are small compared to sonic velocity and also in the case of saml1 temperature decreases (compared with the absolute valur) and, finally, when the molecular weights of the jet gas and the surrounding medium are similar.

It stands to reason that the flow of a gas with $\rho \approx$ const may be obtained as a particular case of the flow of a compressible gas. As to the method of solution and the independent meaning, it is, however, necessary to consider the jets of incompressible fluids separately. The characteristic feature of a flow with $\rho =$ const is, as already mentioned above, the independency of the dynamic problem and the thermal problem, a fact that simplifies the investigation essentially.

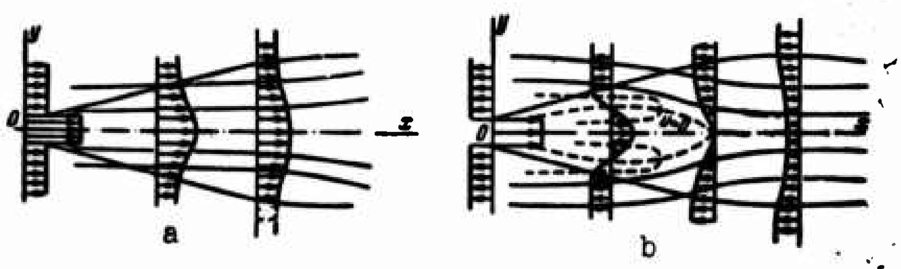

Fig. 0.3. Schematic representation of the expansion of a jet in a flow. a)Flow in the same direction; b)counterflow.

As regards the jets of compressible gases, we must among them distinguish between the manifestations of the compressibility which are due to the influence of one or several factors of the compressibility acting simultaneously, the high velocity of the motion or a considerable difference in temperature or composition between the gas in the jet and the surrounding medium.

The subdivision of the jets into homogeneous in composition (as or liquid) and inhomogeneous ones may be attributed to the same feature (the compressibility). The latter type of jets fall into classes according to the properties of the carrying medium and the nature of the impurities (gas, liquid or solid particles).

The matter in a jet motion may thus be classed in this form or other according to the three states of aggregation. In addition to this, attention has been attracted in the past years by jets of plasma, the fourth state of aggregation of matter.

The subdivision of jets according to the feature of the compressibility or the state of aggregation takes the state of the matter in both the jet and the surrounding space into account. Considering the interactions between jet and surrounding medium in this way, it is expedient to take another two features into account, the conditions of motion of the medium and its geometry.

As to the former we must distinguish between the expansion of a jet in a resting and in a moving medium and in this particular case between a flow in the same direction as the jet, a counterflow or a flow making a certain angle with the jet (see schematic diagram of Fig. 0.3). To this group of problems the flow in a jet boundary layer is related which arises owing to the instability of the tangential discontinuity appearing between two homogeneous semi-infinite flows moving in the same direction or oppsitely (see Fig. 0.4).

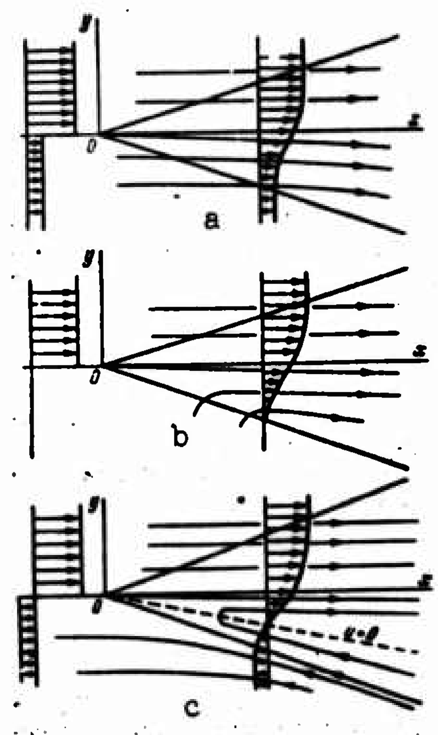

Fig. 0.4. Boundary mixing of parallel flows. a) Flows in the same direction; b) margin of plane jet; c) counterflow.

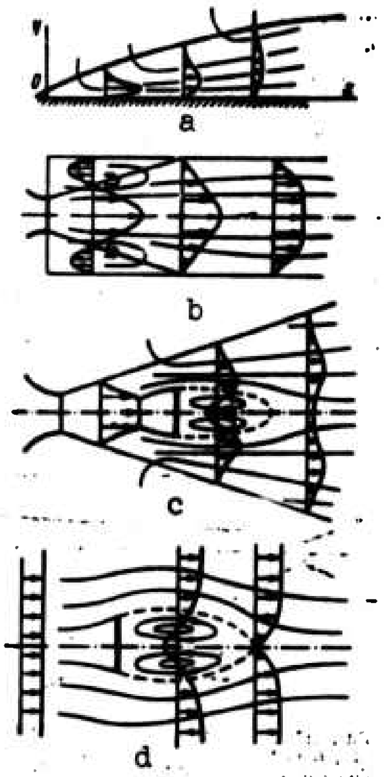

Fig. 0.5. Schematic representations of bound jets. a) Semilimited jet; b) jet in limited space; c) jet flowing around a body; d) homogeneous flow around a body.

As regards the geometry of the space in which the efflux takes place we must consider separately the expansion of a jet in a virtually unbounded space, in the absence of any solid bodies which could interact with the jet; such flows are denoted free jets. In addition to this, we must distinguish "semilimited" jets moving along a solid surface, "bounded" jets, flowing into a limited space and, finally, the motions which occur when bodies of finite dimensions are placed in the jet flow. Figure 0.5 shows some examples of such types of flows.

It is natural that all types of motion discussed above and in the following may be plane-parallel, axisymmetric or, in the general case, threedimensional. The geometry of the nozzle producing the jet is predetermined to an essential degree by this division. Just as the source

jet, the real jets (flowing out of nozzles of finite dimensions) may be plane, axisymmetric, annular, fan-shped, etc., twisted or not, displaying various forms of the initial velocity profile (or temperature distribution, etc.).

Considering the differences in the initial distributions of velocity temberature and other characteristic parameters, we have in fact arrived at another important feature of jet motion systematics, the type of substance transferred. In this sense we distinguish between the dynamic problem, the thermal problem and the diffusion problem. Ofter it is two or even all three types of transfer which coact. In these cases as, e.g., the efflux of a plane-parallel or fan-type jet, it is essential to distinguish between presence and absence of similarities in the velocity and temperature boundary conditions (Fig. 0.6). different boundary conditions for the thermal problem (adiabatically insulated wall, wall of constant temperature, etc.) and for the velocity distribution (motion with or without flow in the same direction, flow along a porous wall, the pores being sinks or sources of additional mass, etc.) also characterize the semilimited jets. These features together with those considered above constitute a great variety of jet flows.

In the individual classes of jet flows we must distinguish motions of liquid and gas and whether there are accompanying chemical reactions or changes in the state of aggregation. Particularly important among them are the jet flows of hot gas, the so-called flares, resulting from the combustion of a gas mixture prepared previously (homogeneous flare) or in the case of a combustion of non mixed gases (diffusion flare).

An independent class of jet flows is formed by the jets of electrically conducting fluids, interacting with an electromagnetic field. Some of these particular cases of jet flows are considered at the end

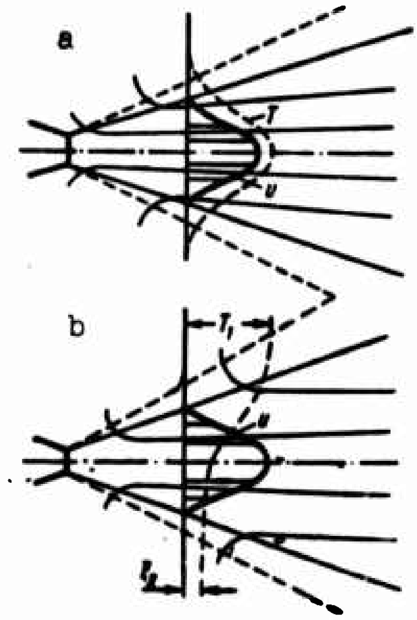

Fig. 0.6. Schmatic representation of a plane nonisothermal jet. a)Symmetrical thermal boundary layer; b)asymmetric thermal boundary layer.

of this book.

REFERENCES*

1, 4, 5, 11, 25, 26, 58, 77, 114, 122, 130, 134, 135, 151, 152, 157, 174, 175, 193, 194, 206, 212, 214, 232, 282, 300.

Part One

SOLUTIONS TO THE NAVIER-STOKES EQUATIONS

In the present part, starting from the Navier-Strokes equations, we shall consider the problem of the expansion of an axisymmetric jet of a viscous incompressible fluid streaming out of a thin tube. The solution is constructed for the fluid flow at at a considerable distance from the nozzle, i.e., at a distance large compared to the dimensions of the orifice. For generality we also consider the thermal (diffusion) problem in addition to the dynamical problem, though the solution to the latter is obtained independently as this is usual for an incompressible fluid with constant physical properties.

The solution of the problem, taking into account the specific features of the flow, the axial symmetry and the ceasing of motion at infinite distance from the nozzle, is built up in the form of series expansions in decreasing powers of the distance from the source. It is shown that the fundamental features of the effect are contained in the so-called self-similar solution. This solution corresponds to the first approximation in the series expansion of the expressions of velocity components, the fluid being free from rotation. In the presence of rotation the approximate self-similar solution may be obtained taking into account the faster decrease of the rotational component of velocity.

In addition to the general investigations we shall consider particular cases of jet flows in axial symmetry, under various boundary condary conditions, the expansion of a jet in an unlimited space or a flow bounded by surfaces.

One of these cases is that of a source jet flowing out of a thin

tube, or the flow inside a cone or a twisted fan-type jet.

We shall point out in a general form the conditions of self-similitude of the motion, the position-dependent inherent dimensions of relations and the intgral characteristics which, when given for a self-similar flow, replace the detailed efflux conditions for the nozzle.

Finally we shall pass over in the solutions obtained from the exact Navier-Stokes equations for a "powerful" jet to a form characteristic of the theory of the laminar boundary layer.

Chapter 1

A JET FLOWING OUT OF A THIN TUBE

1.1. THE PROBLEM

For the group of problems considered here, L.D. Landau's paper [122] in which a self-similar solution was obtained for the direct jet flow is of fundamental importance. A generalization for the thermal (diffusion) problem and also a solution for a jet with nonvanishing angular momentum and a series of particular cases were considered in the papers [116, 162, 163, 171, 203, 302, 303].

Let us study the motion of a fluid produced by a source jet with a directed initial momentum. In the case where the initial momentum is directed along the symmetry axis of the flow a graphic representation* of the motion is given in cylindrical coordinates, r, φ, x. As regards the mathematical treatment of the problem it is, however, simpler to use spherical coordinates, R, θ, φ.

Let us, for convenience, give the formulas which link the coordinates and velocity components in the cylindrical and spherical coordinate systems:

$$r = R\sin\theta, \quad \varphi = \varphi, \quad x = R\cos\theta,$$
$$v_r = v_R \sin\theta + v_\theta \cos\theta, \quad v_\varphi = v_\varphi, \quad v_x = v_R \cos\theta - v_\theta \sin\theta.$$

The velocity components v_r, v_φ, v_x in the cylindrical system of coordinates will be denoted the radial, peripheral and axial components, respectively. The quantities v_R, v_θ, v_φ in the spherical system of coordinates (i.e., the projections of the velocity vector on the

axes R, θ, φ) are not given special denotations to avoid confusion, except for the velocity v_φ which is the same in both systems of coordinates.

The initial system of differential equations of continuity, motion and heat (mass) transfer for a steady axisymmetric flow of an incompressible viscous fluid, in the absence of mass forces, will be written in the spherical system of coordinates R, θ, φ. The origin of coordinates is allowed to coincide with the source of the jet, the polar axis Ox is directed along the axis of symmetry of the flow.

The continuity equation:

$$\frac{\partial v_R}{\partial R} + \frac{1}{R}\frac{\partial v_\theta}{\partial \theta} + \frac{2v_R}{R} + \frac{v_\theta \operatorname{ctg}\theta}{R} = 0. \tag{1.1}$$

The equations of motion:

$$v_R \frac{\partial v_R}{\partial R} + \frac{v_\theta}{R}\frac{\partial v_R}{\partial \theta} - \frac{v_\theta^2 + v_\varphi^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial R} + \nu\left(\frac{\partial^2 v_R}{\partial R^2} + \frac{1}{R^2}\frac{\partial^2 v_R}{\partial \theta^2} + \right.$$
$$\left. + \frac{2}{R}\frac{\partial v_R}{\partial R} + \frac{\operatorname{ctg}\theta}{R^2}\frac{\partial v_R}{\partial \theta} - \frac{2}{R^2}\frac{\partial v_\theta}{\partial \theta} - \frac{2v_R}{R^2} - \frac{2v_\theta}{R^2}\operatorname{ctg}\theta\right), \tag{1.2}$$

$$v_R \frac{\partial v_\theta}{\partial R} + \frac{v_\theta}{R}\frac{\partial v_\theta}{\partial \theta} + \frac{v_R v_\theta}{R} - \frac{v_\varphi^2}{R}\cdot \operatorname{ctg}\theta = -\frac{1}{\rho R}\frac{\partial p}{\partial \theta} +$$
$$+ \nu\left(\frac{\partial^2 v_\theta}{\partial R^2} + \frac{1}{R^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{R}\frac{\partial v_\theta}{\partial R} + \frac{\operatorname{ctg}\theta}{R^2}\frac{\partial v_\theta}{\partial \theta} + \frac{2}{R^2}\frac{\partial v_R}{\partial \theta} - \frac{v_\theta}{R^2 \sin^2\theta}\right), \tag{1.3}$$

$$v_R \frac{\partial v_\varphi}{\partial R} + \frac{v_\theta}{R}\frac{\partial v_\varphi}{\partial \theta} + \frac{v_R v_\varphi}{R} + \frac{v_\theta v_\varphi}{R}\operatorname{ctg}\theta =$$
$$= \nu\left(\frac{\partial^2 v_\varphi}{\partial R^2} + \frac{1}{R^2}\frac{\partial^2 v_\varphi}{\partial \theta^2} + \frac{2}{R}\frac{\partial v_\varphi}{\partial R} + \frac{\operatorname{ctg}\theta}{R^2}\frac{\partial v_\varphi}{\partial \theta} - \frac{v_\varphi}{R^2 \sin^2\theta}\right). \tag{1.4}$$

The equation of heat transfer (neglecting viscous dissipation):

$$v_R \frac{\partial T}{\partial R} + \frac{v_\theta}{R}\frac{\partial T}{\partial \theta} = a\left(\frac{\partial^2 T}{\partial R^2} + \frac{1}{R^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{2}{R}\frac{\partial T}{\partial R} + \frac{\operatorname{ctg}\theta}{R^2}\frac{\partial T}{\partial \theta}\right). \tag{1.5}$$

In these equations v_R, v_θ, v_φ are the velocity components, p, T, and ρ denote pressure, temperature and density, ν and a are the coefficients of kinematic viscosity and thermal diffusivity which (just as the diffusion coefficient D, see below) are considered constant.

Neglecting the diffusion due to temperature and pressure, the dif-

fusion equation analogous to Eg. (1.5) would read

$$v_R \frac{\partial c}{\partial R} + \frac{v_\theta}{R}\frac{\partial c}{\partial \theta} = D\left(\frac{\partial^2 c}{\partial R^2} + \frac{1}{R^2}\frac{\partial^2 c}{\partial \theta^2} + \frac{2}{R}\frac{\partial c}{\partial R} + \frac{\operatorname{ctg}\theta}{R^2}\frac{\partial c}{\partial \theta}\right), \quad (1.5a)$$

c being the concentration and D the diffusion coefficient.

Since the equations of heat transfer (neglecting the heat of friction) and mass transfer are of the some form, their solutions, written in a nondimensional form, will of course, also be identical in the case of similar boundary conditions for temperature and concentration. In the following we shall therefore ignore Eq. (1.5a) and for brevity only speak of heat transfer. This analogy is violated when viscous dissipation (or radiation) is taken into account in the heat transfer equation of when temperature- or pressure-induced diffusion is taken into account in the diffusion equation.

Equation (1.1) does not contain the peripheral velocity v_φ. Let us there introduce a stream function of meridional flow in the form

$$\psi = \nu\left\{R g_1(\theta) + g_2(\theta) + \frac{1}{R} g_3(\theta) + \cdots\right\}. \quad (1.6)$$

The velocity components v_R and v_θ are here given by

$$v_R = \frac{1}{R^2 \sin\theta}\frac{\partial \psi}{\partial \theta}, \quad v_\theta = -\frac{1}{R\sin\theta}\frac{\partial \psi}{\partial R},$$

and defined by the expressions

$$v_R = \frac{\nu}{\sin\theta}\left\{\frac{1}{R}\frac{dg_1}{d\theta} + \frac{1}{R^2}\frac{dg_2}{d\theta} + \cdots\right\}, \quad (1.7)$$

$$v_\theta = -\frac{\nu}{\sin\theta}\left\{\frac{g_1}{R} - \frac{g_3}{R^3} - \cdots\right\}. \quad (1.8)$$

Note that the face that the expansion of the stream function ψ does not contain terms containing the independent variable R in powers higher than the first is due to the conditions of regularity of the velocity

vector components in the entire field of flow (with the exception of the origin of coordinates which is a singularity) and the requirement that the velocity components vanish as $R \to \infty$.

Let us introduce a new indipendent variable

$$\omega = \cos\theta$$

and give the transformation equations in explicit form:

$$\frac{d}{d\theta} = -\sqrt{1-\omega^2}\frac{d}{d\omega},$$
$$\frac{d^2}{d\theta^2} = -\omega\frac{d}{d\omega} + (1-\omega^2)\frac{d^2}{d\omega^2}.$$

The expressions of the velocity components v_R and v_θ, taking the identity $g_i(\theta) \equiv f_i(\omega)$ into account, can be written in the form

$$v_R = -\nu\left\{\frac{f_1'(\omega)}{R} + \frac{f_2'(\omega)}{R^2} + \dots\right\}, \qquad (1.9)$$

$$v_\theta = -\nu\left\{\frac{f_1(\omega)}{R\sqrt{1-\omega^2}} + \dots\right\}. \qquad (1.10)$$

Let us also put

$$v_\varphi = \nu\left\{\frac{\Phi_1(\omega)}{R} + \frac{\Phi_2(\omega)}{R^2} + \dots\right\}, \qquad (1.11)$$

$$\frac{p - p_\infty}{\rho} = \nu^2\left\{\frac{h_1(\omega)}{R} + \frac{h_2(\omega)}{R^2} + \dots\right\}, \qquad (1.12)$$

$$T - T_\infty = \frac{\tau_1(\omega)}{R} + \frac{\tau_2(\omega)}{R^2} + \dots, \qquad (1.13)$$

where p_∞ and T_∞ are the pressure and temperature in a nonperturbed fluid far away from the source, the primed quantities are derivatives with respect to ω. Let us substitute Eqs. (1.9) - (1.13) in Eqs. (1.2) - (1.5) and in the equations obtained put equal to one another the coefficients of equal powers R in the right-hand and left-hand sides of each of these equations. To show what is meant we give by way of example one of the transformed equations (corresponding to Eq. (1.2) for the velocity component v_R):

$$
\begin{aligned}
&-\left(\frac{f_1'}{R}+\frac{f_2'}{R^2}+\cdots\right)\left(\frac{f_1'}{R^3}+2\frac{f_2'}{R^4}+\cdots\right)-\frac{f_1}{R^2}\left(\frac{f_1''}{R}+\frac{f_2''}{R^2}+\cdots\right)-\\
&-\frac{f_1^2}{R^2(1-\omega^2)}-\frac{\Phi_1^2}{R^2}-2\frac{\Phi_1\Phi_2}{R^3}-\cdots=\frac{h_1}{R^2}+2\frac{h_2}{R^3}+3\frac{h_3}{R^4}+\cdots\\
&\cdots-2\frac{f_1'}{R^3}-6\frac{f_2'}{R^4}-\cdots+\frac{\omega f_1''}{R^2}-\frac{1-\omega^2}{R^2}f_1'''+\frac{\omega f_2''}{R^3}-\frac{1-\omega^2}{R^3}f_2'''+\cdots\\
&\cdots+\frac{2}{R^3}f_1'+\frac{4}{R^4}f_2'+\cdots+\omega\left(\frac{f_1''}{R^2}+\frac{f_2''}{R^3}+\cdots\right)-\\
&-\frac{2\sqrt{1-\omega^2}}{R^2}\frac{d}{d\omega}\left[\frac{f_1}{\sqrt{1-\omega^2}}\right]+\frac{2}{R^2}f_1'+\frac{2}{R^3}f_2'+\cdots+\frac{2\omega f_1}{R^2(1-\omega^2)}+\cdots
\end{aligned}
\tag{1.2a}
$$

Comparison of the coefficients for $1/R^2$ yields

$$h_1=0,\qquad h_1'=0. \tag{1.14}$$

This indicates that in the expansion (1.12) the first term, containing $1/R$ is lacking.

In an analogous way, putting equal the coefficients with $1/R^3$, simple transformations yield the following equations:

$$[(1-\omega^2)f_1'']'-f_1f_1''-f_1'^2-\frac{f_1^2}{1-\omega^2}-\Phi_1^2-2h_2=0, \tag{1.15}$$

$$\left(\frac{1}{2}\frac{f_1^2}{1-\omega^2}+f_1'\right)'+\frac{\omega}{1-\omega^2}\Phi_1^2+h_2'=0, \tag{1.16}$$

$$(\Phi_1\sqrt{1-\omega^2})''-\frac{f_1}{1-\omega^2}(\Phi_1\sqrt{1-\omega^2})'=0, \tag{1.17}$$

$$[(1-\omega^2)\tau_1']'=Pr\,(f_1\tau_1)'. \tag{1.18}$$

$Pr = \nu/a$ being Prandtl's number.

Let us finally give the analogous equations obtained from comparing the coefficients for $1/R^4$:

$$[(1-\omega^2)f_2'']'-f_1f_2''-3f_1'f_2'-2\Phi_1\Phi_2-3h_3=0, \tag{1.19}$$

$$2f_2''+\frac{2\omega}{1-\omega^2}\Phi_1\Phi_2+h_3'=0, \tag{1.20}$$

$$(\Phi_2\sqrt{1-\omega^2})''-\frac{f_1}{1-\omega^2}(\Phi_2\sqrt{1-\omega^2})'+\frac{2-f_1'}{1-\omega^2}(\Phi_2\sqrt{1-\omega^2})=0, \tag{1.21}$$

$$[(1-\omega^2)\tau_2']'+2\tau_2=Pr\,[(f_1\tau_2)'+f_1'\tau_2+f_2'\tau_1]. \tag{1.22}$$

The first three of the above equations (1.15)-(1.18) represent a

closed system of nonlinear ordinary differential equations whose integration determines the dynamic problem. As regards the thermal problem, its solution can be reduced to integrating the linear equation (1.18) using the function $f_1(\omega)$ found previously.

As usual the equations for the following terms of expansions (1.19)-(1.22) and the higher approximations are solved after determining the functions they consist of and which are obtained from the preceding approximations. The equations of the second and higher approximations are then linear differential equations.

Let us now turn to the boundary conditions of the problem. In the general form they must display the axial symmetry of the flow.

Along the axis of the jet (with $\theta = 0$) owing to the symmetry conditions (apart from the vanishing of all derivatives with respect to the coordinate φ which is taken into account in the initial conditions) the velocity components v_θ and v_φ and the derivatives with respect to the angle θ of the velocity component v_R, the pressure and the temperature are vanishing.

Thus

$$v_\theta = v_\varphi = 0, \ \frac{\partial v_R}{\partial\theta} = \frac{\partial p}{\partial\theta} = \frac{\partial T}{\partial\theta} = 0 \ \text{for}\ \theta = 0. \qquad (1.23)$$

When seeking a solution we must, in addition to this, take into account that throughout the region of the flow (except for the origin of coordinates which is a singularity) the values of the velocity components, pressure and temperature are finite.

For Eqs. (1.15)-(1.17) the boundary conditions have the form

$$\lim_{\omega\to 1}\left(\frac{f_1}{\sqrt{1-\omega^2}}\right) = 0, \quad \Phi_1(1) = 0. \qquad (1.24)$$

It is trivial that with $\omega = 1$ the derivatives in the conditions (1.23)

vanish since

$$\frac{d}{d\theta} = -\sqrt{1-\omega^2}\frac{d}{d\omega}.$$

Before we pass over to solve a concrete problem we want to point out an essential singularity which is connected with the choice of the expressions for the celocity components, the pressure and temperature, determined by the expansions (1.9)-(1.13) which grow unlimitedly as the distance to the origin of cooridinates decreses. Owing to this singularity each of the solutions obtained will only describe the flow at a considerable distance from the source. The dimension r_0, of the orific of the efflux is taken as the measure of the remoteness. The condition of applicability of the solution is the inequality $R >> r_0$ which, for the solutions, enables us to ignore shape and dimension of the orifice, velocity and temperature distribution in the efflux section of the nozzle, etc.

Thus the problem is essentially solved without initial conditions. With the accepted statement of the problem of the source jet it is in fact impossible to take into account details of initial conditions, retaining an arbitrary number of terms in the expansions; instead of this integral conditions (conservation of momentum flux, heat content, etc.) are used in the solutions.

When choosing the number of terms in the expansions it is necessary also to consider the following. Though as to their absolute values the velocity components and the temperature tend to infinity as R^n with decreasing R, where n is the higher the larger the number of terms in the expansion; the fact that the signs in the series are alternating may improve the solution a little compared to the first approximation.

But as it is rather cumbersome to calculate the higher approximations we must restrict ourselves in the following manily to finding a

first approximation permitting a good description of the physical image and the most essential laws governing the effect.

1.2. INTEGRATION OF THE EQUATION OF A FIRST APPROXIMATION

Let us determine the first nonvanishing expansion terms for the velocity components, the pressure and the temperature. The totality of these expressions will be called the solutions of the problem in a first approximation. When we add to each of these expressions the next terms of the expansion we obtain respectively the second, third, etc. approximations. The fact that the expansions for different velocity components, pressures and temperatures will be gin with terms containing the variable R in different powers, obviously indicates that the corresponding quantities are damped at different degrees as the distance to the source increases. This could have been predicted from dimensionality considerations as shown below.

Let us first of all consider Eq. (1.17) for the function $\Phi_1(\omega)$, entering the expression of the peripheral velocity (1.11).

Integrating twice yields

$$\Phi_1\sqrt{1-\omega^2} = C_1\int_{-1}^{\omega} \exp\left(\int_{-1}^{\omega}\frac{f_1 d\omega}{1-\omega^2}\right)d\omega + C_2. \qquad (1.25)$$

From Eq. (1.25) with $\omega = -1$, taking into account that the function Φ_1 must be limited, we obtain $C_2 = 0$ and with $\omega = 1$ we also find that the constant $C_1 = 0$ since the integrand

$$\exp\int_{-1}^{\omega}\frac{f_1 d\omega}{1-\omega^2}>0$$

throughout the interval of variation of the variable ω.

Thus $\Phi_1 \equiv 0$.

This indicates that in the first-approximation solution the expression of the peripheral velocity begins with a term containing a

higher power of $1/R$.

By virtue of the equality $\Phi_1 \equiv 0$ the function $h_2(\omega)$ may be eliminated from Eqs. (1.15) and (1.16). For this purpose we integrate Eq. (1.16) and determine the value of the function h_2

$$h_2(\omega) = -\frac{1}{2}\frac{f_1^2}{1-\omega^2} - f_1' + B_2 \tag{1.26}$$

which is substituted in Eq. (1.15). After simple transformations we arrive at the following equation determining the function $f_1(\omega)$:

$$[(1-\omega^2)f_1'']' - f_1 f_1' - f_1'^2 + 2f_1' - 2B_2 = 0.$$

A solution of this equation determines the velocity components v_R and v_θ.

Integrating this equation twice yields the equation

$$2(1-\omega^2)f_1' + 4\omega f_1 - f_1^2 - 2(B_2\omega^2 + B_1\omega + B_0) = 0,$$

which with

$$f_1(\omega) = -2(1-\omega^2)\frac{F'(\omega)}{F(\omega)} \tag{1.27}$$

can be represented as the linear differential equation

$$F'' + \frac{B_0 + B_1\omega + B_2\omega^2}{2(1-\omega^2)^2}F = 0. \tag{1.28}$$

This form of equation was obtained by N.A. Slezkin [173] for an axisymmetric flow of an incompressible viscous fluid.

The derivation of a further solution and its agreement with the boundary conditions for a concrete jet problem depends essentially on the choice of the constants of integration B_0, B_1, B_2. In the simplest case where $B_0 = B_1 = B_2 = 0$ we arrive as shown below at L.D. Landau's problem [122, 302].

Let us now turn to Eq. (1.18) to determine the function $\tau_1(\omega)$ which enters Eq. (1.13) for the temperature distribution. Integrating Eq. (1.18) and noting that the constant of integration vanishes by virtue of the boundary conditions (1.23), we can write (taking into ac-

count the connection between the functions f_1 and F established above)

$$\frac{\tau_1'(\omega)}{\tau_1(\omega)} = \mathrm{Pr}\,\frac{f_1(\omega)}{1-\omega^2} = -2\mathrm{Pr}\,\frac{F'(\omega)}{F(\omega)}. \tag{1.29}$$

From the latter equation we obtain

$$\tau_1(\omega) = B\,[F(\omega)]^{-2\mathrm{Pr}}. \tag{1.30}$$

Temperature and pressure can be defined in the same way in terms of the function $f_1(\omega)$. From the physical point of view this means that the temperature distribution, just as the pressure distribution, is determined by the velocity distribution.

In order to solve the problem in a first approximation such that a complete solution is obtained, it would be necessary to determine the first nonvanishing term in the expression of the peripheral velocity component. For this purpose Eq. (1.21) must be integrated, substituting in it the expression of the function $f_1(\omega)$ obtained for concrete conditions. If in this procedure the function $\Phi_2(\omega)$ proved to be equal to zero, one would have to pass over to the equations for the following terms of the expansion.

Note that if $\Phi_2(\omega) \neq 0$ - and it is only this case which will be considered in what follows-the expression for the peripheral velocity v_φ is determined (with given value of the angular momentum M_x relative to the axis of symmetry of the jet), see below, by the function $f_1(\omega)$. Conversely, the velocity components v_R and v_θ (and the radial and axial velocity components v_r and v_x in cylindrical coordinates which are determined unambiguously by the former) are independent of the peripheral velocity. The same holds true for the temperature and the pressure. As we can see from Eqs. (1.19)-(1.22) this independence of the "twist" is also conserved in the second approximation.

It is quite obvious that this perculiarity of the motion is due to the nature of the method of solution which is applicable only at con-

siderable distances from the source.

1.3. L.D. LANDAU'S PROBLEM

We shall now use the results obtained above for a solution (in a first approximation) of the concrete problem of a source jet flowing out of a thin tube into an unbounded space filled with the same fluid. In addition to paper [122] we also consider the results of paper [203] for the peripheral velocity and papers [163, 302] as regards the temperature.

The solution is reduced to determining the distributions of all velocity components, the pressure and the temperature, at a considerable distance from the source, taking into account the influence of the initial momentum on the flow pattern. As to the mathematical side of the problem, we have to integrate sucessively Eqs. (1.28), (1.21) and to calculate in terms of the given function $f_1(\omega)$ the expressions for $h_2(\omega)$ and $\tau_1(\omega)$ with the help of the formulas (1.26) and (1.30).

In this investigation Eq. (1.28) is of fundamental importance as it contains the three arbitrary constants B_0, B_1, B_2. As already mentioned, the choice of the values of these constants corresponds to the desired choice of the concrete conditions of flow. Unfortunately this choice results in "guessing" the needed values of the constants beforehand and then to verify whether the solution obtained corresponds to the statement of the problem. In this connection it is interesting to subject Eq. (1.28) to a detailed investigation [221]. Some particular cases of solution will be given in the following, here we shall only deal with one of them which corresponds to L.D. Landau's Problem.

Let us put $B_0 = B_1 = B_2 = 0$. As we shall see from the solution, the assumption that the three constants of Eq. (1.28) are all vanishing enables us to satisfy all the necessary conditions of the problem. Note that in paper [122] where Eq. (1.28) does not occur, several terms were

also dropped a priori which was justified by a subsequent verification of the solution to satisfy the conditions of the problem.

From Eq. (1.28) with $B_0 = B_1 = B_2 = 0$ we obtain

$$F'' = 0, \quad F = A_1\omega + A_2,$$

and hence

$$f_1(\omega) = \frac{2L(1-\omega^2)}{1-L\omega}. \tag{1.31}$$

The expression obtained for the function $f_1(\omega)$ satisfies the boundary conditions $f_1(\pm 1) = 0$ for arbitrary values of the constant $L = -A_1/A_2$.

Substituting $f_1(\omega)$ in Eq. (1.21) we obtain the following equation:

$$\Phi'' - \frac{2L}{1-L\omega}\Phi' + \frac{2(1-L^2)}{(1-\omega^2)(1-L\omega)^2}\Phi = 0. \tag{1.32}$$

This equation determines the function $\Phi(\omega)$ which is linked with the sought function $\Phi_2(\omega)$ by the equation

$$\Phi = \Phi_2\sqrt{1-\omega^2}. \tag{1.33}$$

A general solution of Eq. (1.32) has the form

$$\Phi = \frac{CL(1+L)}{2}\frac{1-\omega^2}{(1-L\omega)^2} + A_3\frac{L^2}{(1+L)(1-L\omega)^2}\times$$
$$\times\left\{\left(1+\frac{1}{L^2}\right)\omega - \frac{2}{L} + \frac{1}{2}\left(\frac{1}{L^2}-1\right)(1-\omega^2)\ln\frac{1+\omega}{1-\omega}\right\},$$

which yields for the function Φ_2

$$\Phi_2 = C\frac{L(L+1)}{2}\frac{\sqrt{1-\omega^2}}{(1-L\omega)^2} + A_3\frac{L^2}{(1+L)(1-L\omega)^2\sqrt{1-\omega^2}}\times$$
$$\times\left\{\left(1+\frac{1}{L^2}\right)\omega - \frac{2}{L} + \frac{1}{2}\left(\frac{1}{L^2}-1\right)(1-\omega^2)\ln\frac{1+\omega}{1-\omega}\right\}.$$

It is obvious that one has to set $A_3 = 0$ to prevent the function $\Phi_2(\omega)$ from growing unlimitedly as $\omega \to 1$.

Thus

$$\Phi_2 = \frac{CL(1+L)}{2}\frac{\sqrt{1-\omega^2}}{(1-L\omega)^2}, \tag{1.34}$$

C being the constant of integration. The expression (1.34) has been obtained in paper [203] by M.S. Tsukker.

In order to determine the pressure distribution, let us determine

the function $h_2(\omega)$ from Eq. (1.26) with $B_2 = 0$ and the above expression for the function $f_1(\omega)$:

$$h_2(\omega) = -4L\frac{L-\omega}{(1-L\omega)^3}. \qquad (1.35)$$

Thus we see that all functions needed to build up a solution of the dynamic problem in a first approximation can be expressed in terms of two parameters, namely the constants L and C.

Let us now turn to the determination of the function $\tau_1(\omega)$ entering the expression for the temperature.

From Eq. (1.30), taking into account that $F = 1-L\omega$, it follows that

$$\tau_1(\omega) = B(1-L\omega)^{-2P_r}. \qquad (1.36)$$

The expressions (1.31), (1.34), (1.35) and (1.36) furnish a complete solution to the problem in first approximation. They still contain the three constants L, C and B which remain unknown as they cannot be determined from the boundary conditions.

In order to determine the constants L, C, B we need some additional conditions which replace the initial conditions lacking in the problem.

As already mentioned, for solutions which are valid at great distances from the source, the values of $d_0 = 2r_0$, the characteristic dimension (tube diameter), the initial values of velocity, temperature, etc. are unessential. This, of course, does not mean complete arbitrariness in the choice of initial conditions. In a concrete flow problem, in addition to the physical constants of the effluent fluid (ρ, ν, a) it is necessary to know some integral characteristics of the jet with the help of which the constants L, C and B can be found.

REFERENCES

116, 122, 162, 163, 171, 172, 173, 174, 203, 221, 302, 303.

Chapter 2

SOME PROPERTIES OF THE SOLUTION

2.1. REMARKS ON THE FORM OF SOLUTION

When choosing the integral jet characteristics we can, on the basis of dimension considerations, draw some conclusions on the character of Expansions (1.9)-(1.13), etc.

For the first approximation of the problem on an axisymmetric jet (without "twist") in the orifice of the tube with $R \to 0$ the velocity component $v_{x0} \to \infty$, and the initial flow rate of the fluid $G_0 \sim v_{x0} d_0^2 \to 0$ whereas the initial "jet momen[illegible] $J_{x0} \sim v^2_{x0} d_0 2$ is finite (in agreement with the form of solution val[illegible] for $R >> d_0$, $d_0 \sim 0$).

Let us add J_x to the given characteristics of the jet. In this case we can only build up a single expression for the nondimensional velocity components

$$\frac{v_R R}{\nu} = F_R\left(\theta, \frac{J_x}{\rho\nu^2}\right),$$

and

$$\frac{v_\theta R}{\nu} = F_\theta\left(\theta, \frac{J_x}{\rho\nu^2}\right).$$

It is obvious that the components v_R and v_θ (and the axial velocity v_x) are proportional to $1/R$ as Expansions (1.9) and (1.10) begin with this term.

A nondimensional expression for the pressure can be written in the form of

$$\frac{p - p_\infty}{\rho \left(\frac{\nu}{R}\right)^2} = F_p\left(\theta, \frac{J_x}{\rho\nu^2}\right).$$

Therefore

$$p - p_\infty \sim \frac{1}{R^2};$$

the function $h_1(\omega)$ in Eq. (1.12) must be identically equal to zero as this was obtained above.

When J_x is given it is thus possible to construct a solution to the dynamic problem in a first approximation, i.e., to find the constant L.

A nondimensional expression for the temperature can be written in the same approximation in terms of the finite and nonvanishing flux of the surplus heat content $Q \sim v_{x0}(T_0 - T_\infty)d^2_0$. Here

$$\frac{(T - T_\infty)R}{\frac{Q}{\lambda}} = F_T\left(\theta, \frac{J_x}{\rho\nu^2}, \mathrm{Pr}\right),$$

and hence follows

$$T - T_\infty \sim \frac{1}{R},$$

which also agrees with (1.13) ($\lambda = \rho c_p a$ is the coefficient of thermal conductivity). Knowing Q enables us to determine the constant B.

When the rotation is taken into account it is quite natural to choose the quantity of the initial angular momentum flux M_x of the fluid relative to the axis of symmetry O_x as an additional characteristic. The quantity $M_x \sim v_{x0}v_{\varphi 0}d_0{}^3$ will be finite and nonvanishing for $v_x \sim 1/R$ only if $v_\varphi \sim 1/R^2$. The first term of Expansion (1.11) must therefore be identically equal to zero which was already shown when integrating Eq. (1.17) for the function Φ_1.

Thus the nondimentional expression for the peripheral velocity reads

$$\frac{v_\varphi R^2}{\nu R_*} = F_\varphi\left(\theta, \frac{J_x}{\rho\nu^2}\right),$$

$R_* = M_x/J_x$ being a characteristic length.

When M_x is given we can find the constant C. Thus we have solved

our problem in the first approximation considered.

In order to develop the next approximation it is necessary to know an additional quantity of the dimension of a length*. Taking into account that the "addition" to v_R will be proportional to $1/R^2$, we can choose the initial flow rate per second, denoted by G_0, as this quantity, which is nonvanishing in the approximation considered ($G_0 \sim v_{x0} \cdot d^2{}_0$ will be a finite quantity for $v_x \sim 1/R^2$). Knowing G_0 enables us to build up nondimentional expressions for the velocity components, the pressure and the temperature.

Thus, instead of having initial conditions of the jet efflux at our disposal, we can use the four integral characteristics of the jet, J_x, M_x, Q and G_0 (the physical constansts ρ, ν, a and c_p being given).

Note that only a single nondimensional parameter can be derived from the first three dimensional quantities (J_x, M_x and Q) and the phys physical constants which must be given simultaneously in order to finally solve the problem in a first approximation; this parameter which does not contain M_x and Q is the Reynolds number

$$\mathrm{Re} = \frac{v_{x0} d_0}{\nu} \sim \sqrt{\frac{J_x}{\rho \nu^2}}$$

(or, after the introduction of a proportionality factor, $\mathrm{Re} = \frac{2}{\sqrt{\pi}} \sqrt{\frac{J_x}{\rho \nu^2}}$).

Reynolds' number Re is the only characteristic parameter which maintains its value not only in the first approximation but also in all the following ones (provided the motion is free from rotation). With $M_x \neq 0$ in the equation which correspond to higher terms of the expansion (beginning with $1/R^3$) the interaction between the axial and the rotational velocity components becomes effective. Together with dimensionality considerations this results in the possibility of deriving another nondimensional parameter from the characteristics M_x and G_0 (and the constants), a ratio of characteristic lengths. This parameter be-

longs to the arguments for nondimensional pressure and temperature distrivutions in the corresponding approximations. Here, as always in problems on the motions of an incompressible fluid with nonvarying physical properties, the temperature in its turn will not influence the fluid's flow.

2.2. SELF-SIMILAR SOLUTIONS

A solution of the basic system of equations written above in the nondimensional form

$$\left.\begin{aligned} &\frac{v_R R}{\nu} = F_R(\theta, \mathrm{Re}), \quad \frac{v_\theta R}{\nu} = F_\theta(\theta, \mathrm{Re}), \quad \frac{v_\varphi R^2}{\nu R_0} = F_\varphi(\theta, \mathrm{Re}), \\ &\frac{(p-p_\infty) R^2}{\rho \nu^2} = F_p(\theta, \mathrm{Re}), \quad \frac{(T-T_\infty) R}{\frac{Q}{\lambda}} = F_T(\theta, \mathrm{Re}, \mathrm{Pr}), \end{aligned}\right\} \qquad (2.1)$$

displays of the following remarkable peculiarity. When each of the above expressions is written in th form of a ratio, e.g.,

$$\frac{v_R}{v_{R,\theta=0}} = f_R(\theta, \mathrm{Re}), \quad \frac{T-T_\infty}{(T-T_\infty)_{\theta=0}} = f_T(\theta, \mathrm{Re}, \mathrm{Pr}) \text{ etc.}$$

these ratios will not depend on the coordinate R. This indicates that with arbitrary values of R the relative velocity, temperature and pressure distributions will be identical (with any value given for the parameter Re) and the absolute distributions will thus also be similar.

This property of the solutions is conserved in the transition to cylindrical coordinates

$$\left.\begin{aligned} &\frac{v_x}{v_{x|r=0}} = f_x\left(\frac{r}{x}, \mathrm{Re}\right), \quad \frac{v_r}{v_{rm}} = f_r\left(\frac{r}{x}, \mathrm{Re}\right), \quad \frac{v_\varphi}{v_{\varphi m}} = f_\varphi\left(\frac{r}{x}, \mathrm{Re}\right), \\ &\frac{p-p_\infty}{(p-p_\infty)_{r=0}} = f_p\left(\frac{r}{x}, \mathrm{Re}\right), \quad \frac{T-T_\infty}{(T-T_\infty)_{r=0}} = f_T\left(\frac{r}{x}, \mathrm{Re}, \mathrm{Pr}\right), \end{aligned}\right\} \qquad (2.2)$$

where $r/x = \mathrm{arctg}\,\theta$; v_{rm}, $v_{\varphi m}$ and characteristic (e.g., Maximum) values of the velocity components. These solutions (and the corresponding motions of the unbounded medium) are called self-similar.

The solution in first approximation is thus self-similar.

In order to evaluate this result correctly let us turn to a general investigation of the problem of the self-similar solutions of the

Navier-Stokes and energy Equations (1.1)-(1.5)(see [57]).

In a general form the slef-similar solutions (2.1) can be written in terms of the product $F_i(\theta)\cdot R^{-n}$. Let us look for which values of the exponent n self-similar solutions of the basic system of equations may exist. Owing to considerations of generality we cannot restrict ourselves from the very beginning to jet flows alone so that we retain in the solutions the explicit expressions of velocity, pressure and temperature, given in terms of the coordinates R and θ alone (for flows of axial symmetry).

We put

$$\left.\begin{array}{l} v_R=\dfrac{\nu}{R^\alpha}F_R(\theta),\quad v_\theta=\dfrac{\nu}{R^\beta}F_\theta(\theta),\quad v_\varphi=\dfrac{\nu}{R^\gamma}F_\varphi(\theta),\\ \dfrac{p-p_\infty}{\rho}=\nu^2\dfrac{F_p(\theta)}{R^\delta},\quad T-T_\infty=\dfrac{F_T(\theta)}{R^\varepsilon}. \end{array}\right\} \qquad (2.3)$$

Note that instead of these expressions we could have written more general ones, of the form of $v_R = F_R(\theta)V_R(R)$ etc. It can, however, be shown (by substituting these expressions in the basic equations) that this generalization does not yield any new self-similar solutions apart from the power functions ($V_R \sim R^{-n}$).

Substituting Eq. (2.3) in the system of Eqs. (1.1)-(1.5) we obtain

$$\begin{gathered} -\alpha\frac{F_R^2}{R^{2\alpha+1}}+\frac{F_RF_\theta}{R^{\alpha+\beta+1}}-\frac{F_\theta^2}{R^{2\beta+1}}-\frac{F_\varphi^2}{R^{2\gamma+1}}=\\ =\delta\frac{F_p}{R^{\delta+1}}+\frac{1}{R^{\alpha+2}}\{\alpha(\alpha-1)F_R+F_R''+F_R'\operatorname{ctg}\theta-2F_R\}-\\ -\frac{2}{R^{\beta+2}}(F_\theta'+F_\theta\operatorname{ctg}\theta),\\ (1-\beta)\frac{F_RF_\theta}{R^{\alpha+\beta+1}}+\frac{F_\theta F_\theta'}{R^{2\beta+1}}-\frac{F_\varphi^2}{R^{2\gamma+1}}\operatorname{ctg}\theta=\\ =-\frac{F_p'}{R^{\delta+1}}+\frac{1}{R^{\beta+2}}\left\{\beta(\beta-1)F_\theta+F_\theta''+F_\theta'\operatorname{ctg}\theta-\frac{2F_\theta}{\sin^2\theta}\right\}+\frac{2F_R'}{R^{\alpha+2}}, \qquad (2.4)\\ (1-\gamma)\frac{F_RF_\varphi}{R^{\alpha+\gamma+1}}+\frac{F_\theta}{R^{\beta+\gamma+1}}(F_\varphi'+F_\varphi\operatorname{ctg}\theta)=\\ =\frac{1}{R^{\gamma+2}}\left\{\gamma(\gamma-1)F_\varphi+F_\varphi''+F_\varphi\operatorname{ctg}\theta-\frac{F_\varphi}{\sin^2\theta}\right\},\\ (2-\alpha)\frac{F_R}{R^{\alpha+1}}+\frac{1}{R^{\beta+1}}(F_\theta'+F_\theta\operatorname{ctg}\theta)=0,\\ F_T''+(\operatorname{ctg}\theta-\Pr F_\theta)F_T'+[\varepsilon(\varepsilon-1)+\varepsilon\Pr F_R]F_T=0 \end{gathered}$$

(the prime marks derivatives with respect to θ).

The system obtained goes over to a system of ordinary differential equations if a series of conditions resulting from the requirement that all equations of (2.4) must be independent of the coordinate R, are imposed on the constants α, β, γ, δ (the so-called constants of self-similarity). These conditions are represented by the equalities

$$2\alpha+1=\alpha+\beta+1=2\beta+1=2\gamma+1=\delta+1=\alpha+2=\beta+2,$$
$$\alpha+\gamma+1=\beta+\gamma+1=\gamma+2.$$

The value of the temperature constant ε of self-similarity not been determined from the equations.

A solution to the latter system of linear algebraic equations which (together with the integral conditions of conservation of momentum flux, angular momentum, and excessive heat content, see below) determines the problem unambiguously, has the form

$$\alpha=\beta=\gamma=1, \quad \delta=2, \quad \varepsilon=1. \tag{2.5}$$

The initial system of Eqs. (1.1)-(1.5) will therefore assume the form

$$\left.\begin{aligned}
-F_R^2+F_R'F_\theta-F_\theta^2-F_\varphi^2=2F_p+F_R''+&\\
+F_R'\operatorname{ctg}\theta-2F_\theta'-2F_R-2F_\theta\operatorname{ctg}\theta,&\\
F_\theta F_\theta'-F_\varphi^2\operatorname{ctg}\theta=-F_p'+F_\theta''+F_\theta'\operatorname{ctg}\theta+2F_R'-\frac{F_\theta}{\sin^2\theta},&\\
F_\theta F_\varphi'+F_\theta F_\varphi\operatorname{ctg}\theta=F_\varphi''+F_\varphi'\operatorname{ctg}\theta-\frac{F_\varphi}{\sin^2\theta},&\\
F_R+F_\theta'+F_\theta\operatorname{ctg}\theta=0,&\\
F_T''+(\operatorname{ctg}\theta-\Pr F_\theta)F_T'+\Pr F_RF_T=0.&
\end{aligned}\right\} \tag{2.6}$$

A self-similar solution to this tytsem agreement with the obtained values of the constants of self- similarity must be written the form

$$\left.\begin{aligned}
v_R=\nu\frac{F_R(\theta)}{R},\quad v_\theta=\nu\frac{F_\theta(\theta)}{R},\quad v_\varphi=\nu\frac{F_\varphi(\theta)}{R},&\\
\frac{p-p_\infty}{\rho}=\nu^2\frac{F_p(\theta)}{R^2},\quad T-T_\infty=\frac{F_T(\theta)}{R}.&
\end{aligned}\right\} \tag{2.7}$$

As far as we know this solution has not been found so far for an axisymmetric flow of a viscous fluid. For a more precise statement of the problem, a source jet with zero angular momentum relative to the axis

of symmetry ($M_x \equiv 0$), the result obtained is reduced to the problem considered by L.D. Landau.

At the same time for a twisted jet the result obtained $v_\varphi \sim 1/R$ disagrees with the expression $v_\varphi \sim 1/R^2$ derived above (2.1). In our problem the point is the following.

The system of expressions (2.7) in which $v_\varphi \sim 1/R$ is the only form of self-similar solution for the axisymmetric motion. For a twisted source jet, however, this self-similar solution would correspond to the trivial case of $v_\varphi \equiv 0$ throughout the region of flow since (as shown in the previous section) with $v_r \sim v_\theta \sim v_\varphi \sim 1/R$ the initial angular momentum flux of the source jet vanishes.

It is therefore impossible to apply the self-similar solution (2.7) to the problem of the twisted jet.

On the other hand, maintaining the relationship $v_R \sim v_\theta \sim 1/R$ and assuming $v_\varphi \sim 1/R^2$ we arrive at a nonzero (and finite angular momentum M_x and a self-similar solution of the problem (in a first approximation) in which the initial system of Eqs. (1.1)-(1.5) has been simplified. This consisted in omitting in Eqs. (1.2) and (1.3) the terms containing v^2_φ . This neglection of terms proportional to R^{-5} with respect to the other terms (of the order of R^{-3}) in Eqs. (1.2.) and (1.3) is obviously permissible at great distances from the source. An analogous method was used in the papers [54, 131, 197] etc. in order to obtain self-similar solutions of jet problems within the framework of the boundary layer theory. Such flows are usually denoted "slightly twisted" ones.

It stands to reason that a neglection of individual terms in the initial system of equations and the assumption that the constants of self-similarity are equal to

$$\alpha = \beta = \varepsilon = 1, \quad \gamma = \delta = 2 \qquad (2.8)$$

yields to another system of ordinary equations which hold true instead of Conditions (2.5) and do not coincide with Eqs. (2.6):

$$\left.\begin{aligned}
-F_R^2 + F_R' F_\theta - F_\theta^2 = 2F_p + F_R'' + \\
+ F_R' \operatorname{ctg}\theta - 2F_\theta' - 2F_R - 2F_\theta \operatorname{ctg}\theta, \\
F_\theta F_\theta' = -F_p' + F_\theta'' + F_\theta' \operatorname{ctg}\theta + 2F_R' - \frac{F_\theta}{\sin^2\theta}, \\
F_\theta F_\varphi' + F_\theta F_\varphi \operatorname{ctg}\theta = F_\varphi'' + F_\varphi' \operatorname{ctg}\theta - \frac{F_\varphi}{\sin^2\theta}, \\
F_R + F_\theta' + F_\theta \operatorname{ctg}\theta = 0, \\
F_T'' + (\operatorname{ctg}\theta - \Pr F_\theta) F_T' + \Pr F_R F_T = 0.
\end{aligned}\right\} \qquad (2.9)$$

When we pass over to the independent variable $\omega = \cos\theta$ this system coincides with Eqs. (1.15), (1.16), (1.18) and (1.21) if the function $\Phi_1 \equiv 0$ is used in them.

The self-similar solutions are of great importance in hydrodynamics [1υ8, 177] and, in particular, in the theory of jet motions of a viscous fluid. From the mathematic 1 point of view they are characterized by the relative simplicity of the pertinent ordinary differential equations compared to partial differential equations. This is particu- larly important for the nonlinear problems often encountered in hydrodynamics.

As to the physical point of view, the self-similarity of solutions (that is, similarity of the velocity, pressure and temperature distributions) corresponds to a property which is common to all processes of "leveling" (of velocity, pressure, temperature, potential, etc.), the fact that at a sufficiently large distance from the source of perturbation the initial conditions have no influence. From this point of view the jet flow considered, which is assumed produced by a source jet, can be attributed likewise to the efflux from a thin tube of circular, square, triangular or any other form of cross section.

As regards the estimation of the distance at which the self-similar solution (or a solution obtained by the method of series expansion in increasing powers of $1/R$) describes the actual flow in a suffi-

ciently good approximation, it may only be determined from an accurate (or an approximate, e.g., numerical) solution of the problem of a jet issued from a nozzle of finite dimensions where the concrete initial conditions have been taken into account. Without such a solution we only have the inequality $R \gg d_0$ at our disposal where d_0 is the characteristic dimension of the outlet cross section of the nozzle.

It is obvious that a self-similar solution of the problem considered could have been obtained immediately from the values of the constants of self-similarity (2.8) determined above and from Eqs. (2.9). The way of representation chosen in this chapter can be explained by the wish of considering a more general method of solution by series expansions which enables us to obtain a solution which is somewhat more accurate than the self-similar one.

Since in the following chapters we shall consider solutions of the boundary layer equations, we want to remark in this connection that precisely the above analysis of the system of Navier-Stokes equations for a plane-parallel motion of a viscous fluid results in the absence of a self-similar solution for the analogous problem of a plane source jet. In fact (see [57]), for a self-similar solution to the problem of a plane flow the corresponding constants of self-similarity are $\alpha = \beta = 1$, $\delta = 2$ in formulas of the form

$$v_R \sim \frac{1}{R^\alpha}, \quad v_\theta \sim \frac{1}{R^\beta}, \quad p - p_\infty \sim \frac{1}{R^\delta}$$

(in a polar system of coordinates). Taking the equation of a continuity into account, the equation v_θ = const/R corresponds to these values of α, β and δ, which does not describe a flow produced by a plane and free source jet.*

Having made these remarks we want to turn to the main problem of this chapter, the axisymmetric source jet and its final solution, determining the constants L, C and B.

2.3 THE INTEGRAL CHARACTERISTICS OF THE JET

According to [122] we use the condition of momentum flux conservation in order to determine the constant L. For this purpose we calculate the integral of momentum flux density over a spherical surface of arbitrary radius centered at the source. It is obvious that only one component of the momentum flux tensor, Π_{RR} contributes to this integral; this component is obtained by projecting the momentum flux through a plane whose normal is orientel along R on the axis R.

Taking into account the general relationship between the components of the momentum flux density tensor Π_{ik} and the stress tensor σ_{ik} [122]

$$\Pi_{ik} = \rho v_i v_k - \sigma_{ik},$$

where

$$\sigma_{ik} = -p\delta_{ik} + \mu\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right),$$

we obtain

$$\Pi_{RR} = p + \rho v_R^2 - 2\mu\frac{\partial v_R}{\partial R}$$

and in the projection to the O_x axis

$$J_x = \oint \Pi_{RR}\cos\theta\, ds,$$

where $ds = R^2 \sin\theta d\theta d_\varphi$ is a spherical surface element.

The result of caluslating this integral, taking the above expressions the the pressure and the velocity v_R into account, reads as follows:

$$\bar{J}_x = \frac{J_x}{16\pi\rho\nu^2} = \frac{1}{L} + \frac{4}{3}\frac{L}{1-L^2} - \frac{1}{2L^2}\ln\frac{1+L}{1-L}. \qquad (2.10)$$

For brevity we shall call the quantity J_x (and accordingly $\bar{J}_x$) the "jet momentum".

Equation (2.10) determines the constant L in terms of the initial source-jet characteristic J_x. As to its physical meaning $\bar{J}_x$ corresponds

as indicated above to the square of the characteristic Reynolds number of the source jet

$$\bar{J}_x = \frac{J_x}{16\pi\rho\nu^2} = \frac{\mathrm{Re}^2}{64}$$

or

$$\mathrm{Re} = 8\sqrt{\bar{J}_x} = 8\left(\frac{1}{L} + \frac{4}{3}\frac{L}{1-L^2} - \frac{1}{2L^2}\ln\frac{1+L}{1-L}\right)^{1/2}. \qquad (2.11)$$

To illustrate this relation we give a graph of the function $L = L(\mathrm{Re})$ in Fig. 2.1.

We see that low values of the constant L correspond to low values of the Reynolds number, whereas when Re goes from zero to infinity the L goes from zero to unity.

Though it is well-known that as soon as with a value of Re $\sim$ 15 [224] (corresponding to $L \approx 0.82$) the free axisymmetric laminar jet loses its stability at a certain distance from the nozzle and the flow becomes turbulent we are, with regard to the following, particularly interested in a solution to the problem with the parameter L close to unity. In this case the flow will be characteristic of a problem of the boundary-layer theory of a source jet. This problem will be discussed later on on greater detail.

Let us now determine the constant B of integration from the con - dition of conservation of the flux of excessive heat content Q. Just as in the case of the momentum flux, theqquantity $Q_R \equiv Q$ is obtained as the integral over a spherical surface of the radial component of the heat flux density.*

$$Q = \oint \rho c_p\left[v_R(T - T_\infty) - a\frac{\partial T}{\partial R}\right]ds.$$

Substituting the values of the functions v_R and T-T_∞ from the expression

$$v_R = -\nu\frac{f_1'(\omega)}{R}, \quad T - T_\infty = \frac{\tau_1(\omega)}{R} = \frac{B(1-L\omega)^{-2P_r}}{R},$$

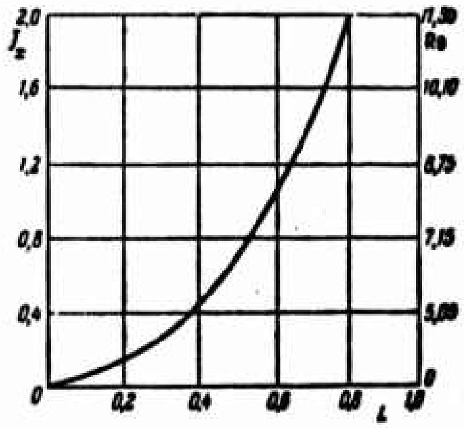

Fig. 2.1. The parameter L as a function of the Reynolds number Re.

we obtain

$$Q = 2\pi\lambda \int_{-1}^{+1} (1 - \mathrm{Pr}\, f_1)\, \tau_1\, d\omega.$$

As the result of these calculations we have

$$B = \frac{Q}{2\pi\lambda} \overline{B}(L, \mathrm{Pr}), \tag{2.12}$$

where

$$\overline{B}(L, \mathrm{Pr}) = \left(\frac{1}{L} - 1\right)^{2\mathrm{Pr}-1} \left\{\frac{2\,\mathrm{Pr}}{2\,\mathrm{Pr}+1} \frac{1+L}{1-L} \left[1 - \left(\frac{1-L}{1+L}\right)^{2\mathrm{Pr}+1}\right] + \right.$$
$$\left. + \left(\frac{1-L}{1+L}\right)^{2\mathrm{Pr}-1} - 1\right\}^{-1}.$$

A graphic representation of the function $\overline{B} = \overline{B}(L, \mathrm{Pr})$ is given in Fig. 2.2. As we see from it, the value of $\overline{B}$ is essentially higher than unity with low values of L. In this region the influence of Prandtl's number is algo considerable. With a value of L of about $L \geq 0.5$ the quantity $\overline{B} < 1$ and $\overline{B} \to 0$ as $L \to 1$.

In order to determine the constant C which enters Expression (1.34) of v_φ we use the condition of conservation of angular momentum flux relative to the axis of symmetry of the jet through a sphere of arbitrary radius R.

The latter may be represented in the form

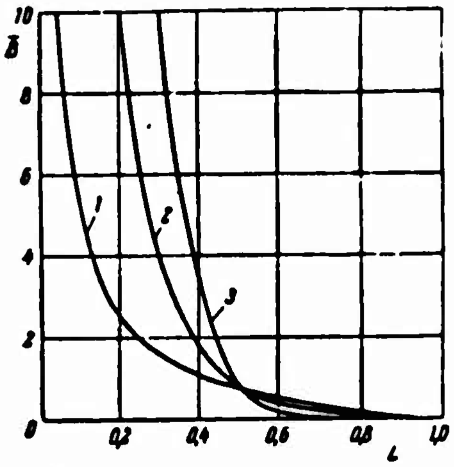

Fig. 2.2. The parameter $\bar{B}$ as dependent on the parameter L and the Prandtl number Pr: 1. Pr = 0.5; 2. Pr = 1.0; 3. Pr = 2.0.

The latter may be represented in the form

$$M_x = 2\pi R^3 \int_0^\pi \Pi_{R\varphi} \sin^2\theta \, d\theta, \qquad (2.13)$$

where the component $\Pi_{R\varphi}$ of the momentum flux density tensor is given by

$$\Pi_{R\varphi} = \rho v_R v_\varphi + \mu\left(\frac{v_\varphi}{R} - \frac{\partial v_\varphi}{\partial R}\right).$$

Determining the velocity components $v_R = -\nu f'_1(\omega)/R$ and $v_\varphi = \nu \Phi_2(\omega)/R^2$ with the help of Eqs. (1.31) and (1.34) we can obtain an expression of $\Pi_{R\varphi}$ and calculate the integral (2.13). The results of these calculations yields the following relation between the angular momentum flux and the constant C:

$$C = \frac{3M_x}{4\pi\rho\nu^2}\bar{C}(L), \qquad (2.14)$$

where

$$\bar{C} = \frac{L(1-L)}{5L^2-3}\left\{1 + \frac{3}{2L}\frac{1-L^2}{5L^2-3}\ln\frac{1+L}{1-L}\right\}^{-1}. \qquad (2.15)$$

Let us give a brief summary.

The expressions for the velocity components, the pressure and temperature obtained above in a first approximation read in their final form:

$$v_R = -\frac{2\nu L}{R}\frac{L-2\omega+L\omega^2}{(1-L\omega)^2}, \tag{2.16}$$

$$v_\theta = -\frac{2\nu L}{R}\frac{\sqrt{1-\omega^2}}{1-L\omega}, \tag{2.17}$$

$$v_\varphi = \frac{C\nu L(1+L)}{2R^2}\frac{\sqrt{1-\omega^2}}{(1-L\omega)^2}, \tag{2.18}$$

$$\frac{p-p_\infty}{\rho} = -\frac{4\nu^2 L}{R^2}\frac{L-\omega}{(1-L\omega)^2}, \tag{2.19}$$

$$T-T_\infty = \frac{B}{R}(1-L\omega)^{-2Pr}. \tag{2.20}$$

To solve the problem in a second approximation the function $h_3(\omega)$ must be eliminated from Eqs. (1.19) and (1.20), an easy task when we take into account that $\Phi_1 \equiv 0$. The equation obtained for the function $F_2 = \frac{1}{C_3} f^2$,

$$[(1-\omega^2)F_2']' - f_1 F_2' + 3(2-f_1)F_2 = 1$$

having substituted in it the expression for the f_1 from (1.31), coincides with Eq. (10) of Paper [162] by Yu.B. Rumer in which its solution was achieved. This solution (and also the solution for the peripheral velocity component and the temperature) and to a still higher degree the further approximations are connected with very cumbersome computations and are therefore not considered here.

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57, 114, 116, 122, 162, 163, 168, 172, 173, 174, 203, 221, 224, 225, 302, 303.

Chapter 3

RESULTS OF THE SOLUTION

3.1. THE FLOW PATTERN

In order to represent the results obtained in a more illustrative form we rewrite the expressions determining the velocity, pressure and temperature distributions in cylindrical coordinates:

$$\left.\begin{aligned}
v_x &= \frac{2\nu L}{x\sqrt{1+\eta^2}}\left\{1+\frac{1-L^2}{(\sqrt{1+\eta^2}-L)^2}\right\},\\
v_r &= \frac{2\nu L\eta}{x\sqrt{1+\eta^2}}\,\frac{1-L\sqrt{1+\eta^2}}{(\sqrt{1+\eta^2}-L)^2},\\
v_\varphi &= \frac{C\nu L(1+L)}{2x^2\sqrt{1+\eta^2}}\,\frac{\eta}{(\sqrt{1+\eta^2}-L)^2},\\
\frac{p-p_\infty}{\rho} &= \frac{4\nu^2 L}{x^2\sqrt{1+\eta^2}}\,\frac{1-L\sqrt{1+\eta^2}}{(\sqrt{1+\eta^2}-L)^2},\\
T-T_\infty &= \frac{B}{x\sqrt{1+\eta^2}}\left(1-\frac{L}{\sqrt{1+\eta^2}}\right)^{-2Pr}.
\end{aligned}\right\} \qquad (3.1)$$

where $\eta = \frac{r}{x}$.

Let us give a few examples to illustrate the character of the flow determined by Eqs. (3.1).

Since Expressions (3.1) are given in terms of the nondimensional parameter L which is an unambiguous function of the characteristic Reynolds number $Re = \frac{2}{\sqrt{\pi}}\sqrt{\frac{J_x}{\rho\nu^2}}$ it is first of all expedient to determine the influence of the latter on the spatial distribution of the variable sought. In this connection we must not forget that the function $L = L(Re)$ represented in Fig. 2.1 shows that when Re is allowed to go from zero to unity. Low values of L will therefore correspond to a jet flow with small momentum flux whereas with $L \to 1$ the flow is produced by a considerable initial momentum.

Following L.D. Landau [122] we shall agree in calling these flows

"weak" and "strong", respectively (bearing mind that in the case of high values of Re and therefore values of L close to unity the actual flow in the jet will be turbulent).

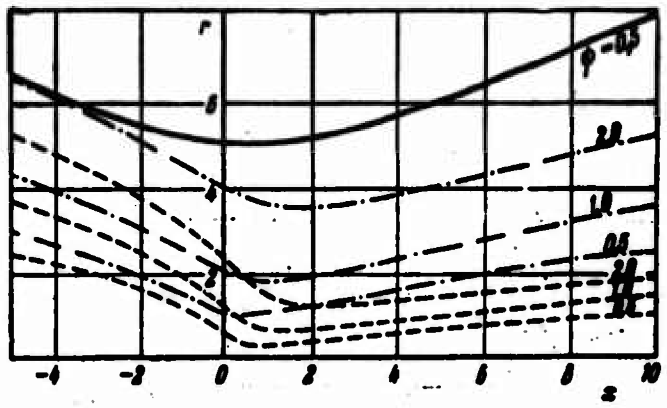

Fig. 3.1. Streamlines in axisymmetric jet.—— $L = 0.1$, "weak" jet; -.-.-. $L = 0.5$; ---- $L = 0.8$, "strong" jet.

A general representation of the flow pattern with different values of L is given by the graph in Fig. 3.1, the streamlines being calculated according to the formula

$$\psi = \nu R g_1(\theta) = 2\nu L \frac{r^2}{\sqrt{r^2+z^2}-Lz}. \qquad (3.2)$$

As we see from the figure the flow region corresponding to a certain given value of ψ = const narrows as L increases, particularly strongly below the source. This indicates that as L increases the whole distinct field of flow assumes the character of a jet, displaying anisotropy of the motion. Precisely this property of the jet is illustrated clearly by the distributions of the relative velocity components $\frac{v_z}{v_{zmax}}$ and $\frac{v_r}{v_{rmax}}$, represented in Fig. 3.2. As L increases, the axial velocity in the cross section of the jet decreases more and more rapidly which is accompanied by a drop of the ratio between radial and axial velocity components. In this connection it must also be taken into account that, as L increases, the damping intensity of the jet decreases with increasing distance from the source as this can be seen from the formula

$$\frac{x v_{x\,max}}{\nu} = \frac{4L}{1-L}$$ (see Fig.3.3).

Figure 3.2 also shows the dependence on L of the characteristic values of the nondimensional coordinate $\eta_* = \frac{r}{x}$, which correspond to the two values $\frac{v_x}{v_{x\,max}} = 0.5$ and 0.1. These curves characterize the conditional width of the zone of perturbations caused by the source jet for several values of the parameter L..Note that the limiting case of a "weak" jet, $L \approx 0$, corresponds to a quite definite finite width of the jet whereas the width of the jet becomes infinitesimal as $L \to 1$.

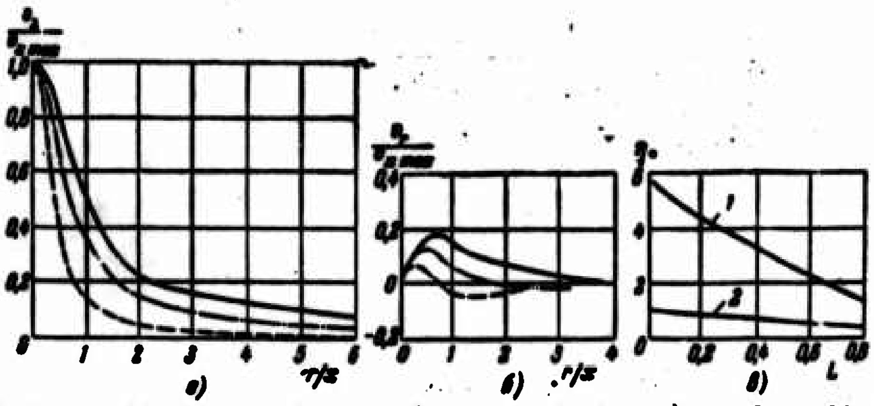

Fig. 3.2. Relative distributions of the axial. a) and radial; b) velocity components in the cross sections of an axisymmetric jet.---- L = 0.1; -.-.-.- L = 0.5; ----- L = 0.8; c) conditional width of the jet η^* as a function of the parameter L. 1. $(v_x/v_{xm}) = 0.1$; 2. $(v_x/v_{xm}) = 0.5$.

For a given value of L the distributions of the relative velocity components are self-similar, i.e., they depend only on the ratio $\eta = \frac{r}{x}$ (see Fig. 3.2). Fig. 3.4 the same distribution $\frac{v_x}{v_{x\,max}}$ is represented as a function of the reduced coordinate $\eta/\eta_{1/2} = r/r_{1/2}$, where $\eta_{1/2}$ corresponds to the value of $\frac{v_x}{v_{x\,max}} = \frac{1}{2}$. This method of representation is often used in the theory of jets.

We see from Fig. 3.4 that, generally speaking, the different distributions of $\frac{v_x}{v_{x\,max}}$, corresponding to different values of L tend to coincide as L increases. Coincidence is virtually reached even with $L \geq 0.8$. This fact will be explained in the next section of this chapter, when we discuss the problem of the properties of a solution with high

values of L.

Finally, Fig. 3.5 shows two sets of values of the velocity v_x and the ratio $\frac{v_x}{v_{x\,max}}$. The shape of these curves is different as we can see from the figure: the lines of equal velocity with respect to the relative velocity are straight lines whereas these lines with respect to v_x are curves join the axis of flow.

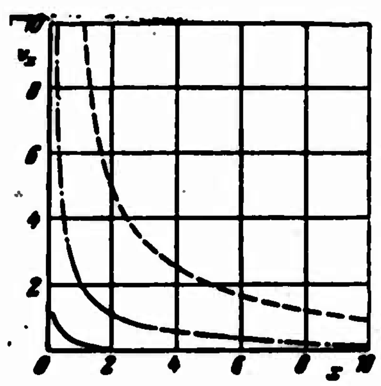

Fig. 3.3. Variation of the axial velocity component along the axis of t the jet (in conventional units). ——— $L = 0.1$; -.-.-.- $L = 0.5$; -------- $L = 0.8$.

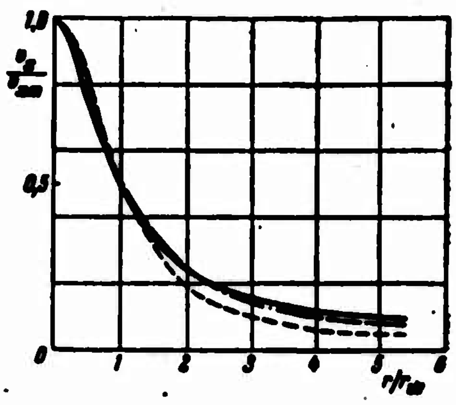

Fig. 3.4. Distribution of axial velocity component in cross sections of the jet. ——— $L = 0.1$; -.-.-.- $L = 0.5$; --- $L = 0.8$.

Figure 3.6a shows the relative distributions of the peripheral velocity component approaches the jet axis as the parameter L increases. There (Fig. 3.6b), in the same cross section (at a certain distance from the source) we choose the value of $v_{x\,max}$ as the scale parameter

of v_{φ}. The figure shows that the parameters L and M_x exert an opposite influence on the intensity of "twisting" and the distribution of $v_{\varphi max}$ in the cross section of the jet.

The jet properties of the flow influence the temperature distributions analogously as the eynamic picture. In addition to the above characteristics of the jet the Prandtl number will also characterize the thermal problem.

Figures 3.7-3.9 show graphs illustrating the decrease of the excessive temperature in the cross sections of the jet and along its axis for several values of the parameters L and Pr. Figure 3.10 shows the presence of a self-similar distribution of the excessive

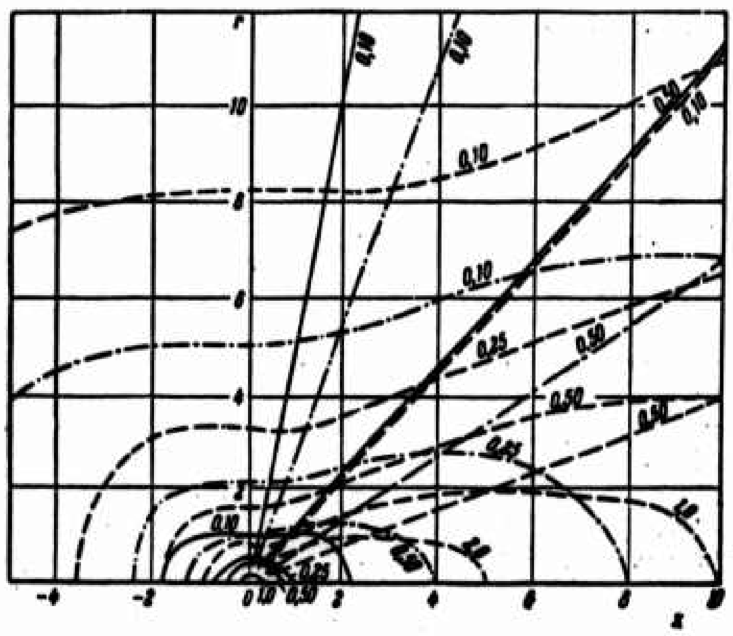

Fig. 3.5. Lines of equal absolute and relative (straight lines) values of the axial velocity component. ——— $L = 0.1$; —·—·— $L = 0.5$; – – – – $L = 0.8$.

temperature for a "strong" jet and the relative courses of the velocity and temperature distributions for several Prandtl numbers. In the case of Pr = 1 the relative velocity and temperature distributions coincide as this is usual in problems of the boundary layer theory to which, essentially, the "strong" jet pertain. With other values of the Prandtl number the temperature distribution curves are correspondingly "broader"

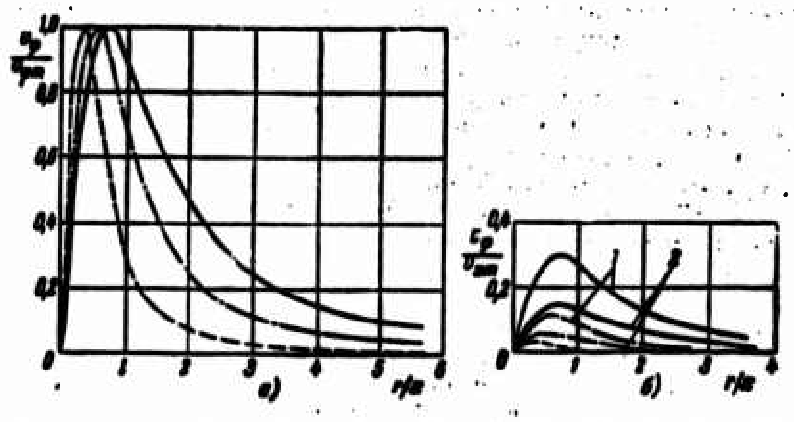

Fig. 3.6. Relative peripheral velocity distributions in jet cross sections. ——— $L = 0,1$; —·—·— $L = 0,5$; – – – – $L = 0,8$; 1. $M_x = 2$; 2. $M_x = 1$.

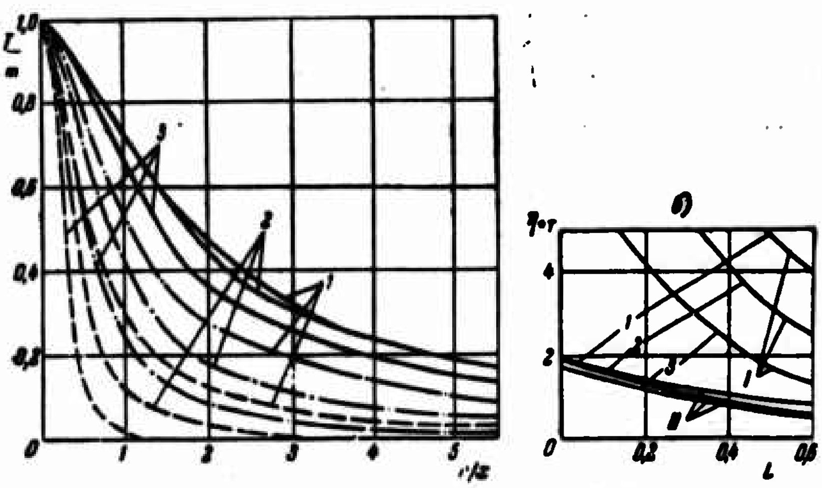

Fig. 3.7. a) Cross-sectional distribution of excessive temperature; b) effective thickness of thermal boundary layer of the jet. I — $\Delta T/\Delta T_m = 0,1$; II — $\Delta T/\Delta T_m = 0,5$; 1. $Pr = 0,5$; 2. $Pr = 1,0$; 3. $Pr = 2,0$. ——— $L = 0,1$; —·—·— $L = 0,5$; – – – – – $L = 0,8$.

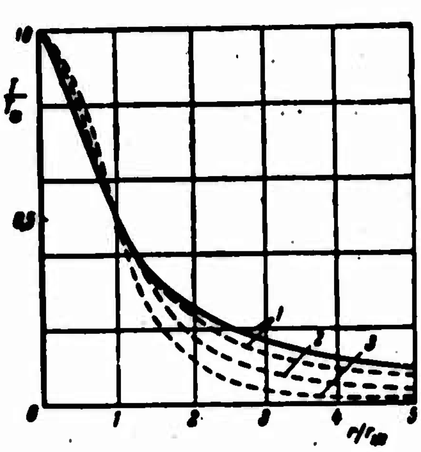

Fig. 3.8. Cross-sectional distributions of excessive temperature. 1. $Pr = 0,5$; 2. $Pr = 1,0$; 3. $Pr = 2,0$. ——— $L = 0,1$; – – – – $L = 0,8$.

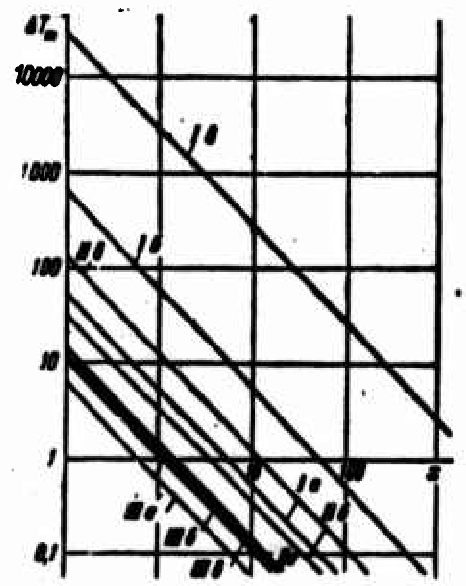

Fig. 3.9. Variation of excessive temperature along the jet axis (in conventional units). I — L = 0,1; II — L = 0,5; III — L=0,8; а. Pr = 0,5; б. Pr=1,0; в. Pr=2,0.

than the velocity distributions if Pr < 1 or "narrower" if Pr > 1. In other words, in a case where, e.g., the mentum transfer coefficient is smaller than the heat transfer coefficient ($\nu < a$, Pr < 1) the temperature drops much more rapidly along the jet axis than the velocity; at the same time the width of the thermal "trace" (the region of temperature perturbation) in the same cross sections will exceed the width of the dynamic "trace". The smaller Pr the broader will the heat layer be where the influence of Prandtl's number Pr is great in the marginal parts of the jet and relatively small in the middle (compart the η^* curve for $\frac{\Delta T}{\Delta T_m} = 0{,}1$ and 0.5 in Fig. 3.7b).

The relationships between the dynamic and thermal properties of the jet derived above the general, considered from a qualitative point of view, they are the same in all jet flows, including the problem of the laminar and turbulent coundary layer jet.

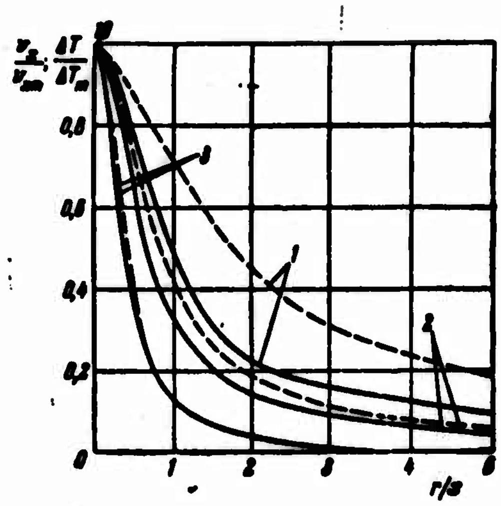

Fig. 3.10. Relative distributions of velocity (solid lines) and temperature in cross section of the jet (Pr - 1). 1. $L = 0.1$; 2. $L = 0.5$; 3. $L = 0.8$.

3.2. FLOW INSIDE A CONE

Let us briefly discuss the flow within a right circular cone on the basis of Eqs. (1.28) and (1.29) as another example for the motion of a viscous fluid which, in its properties, is similar to the jet flow. The problem can be sketched as follows. We assume the initial momentum flux of the jet, issued from a small orifice in the vertex, oriented along the axis of the cone; the lateral surface of the latter is taken as the surface of the stream. In other words, the problem of the expansion of an untwisted jet of a viscous fluid in a bounded space will be solved under boundary conditions corresponding to a perfect fluid*. From the general solution of this problem [116] we obtain as a particular case the solution of the problem of a jet issued from an opening in a plane wall, perpendicularly to the latter [303].

We shall consider the dynamic and the thermal (diffusion) problem at the same time.

As in L.D. Landau's problem, we satisfy the conditions chosen by an appropriate choice of the constants B_0, B_1 and B_2 in Eq. (1.28), considering the self-similar solutions of the form of

$$\left.\begin{aligned} v_R &= -\nu \frac{f_1'(\omega)}{R}, \quad v_\theta = -\nu \frac{f_1(\omega)}{\sqrt{1-\omega^2}}, \\ \frac{p-p_\infty}{\rho} &= \nu^2 \frac{h_0(\omega)}{R^2}, \quad T - T_\infty = \frac{\tau_1(\omega)}{R}. \end{aligned}\right\} \tag{3.3}$$

For this problem we have

$$B_0 = B_2 = -\frac{1}{2} B_1 = \frac{4b^2+1}{2}.$$

In this case, instead of the three constants B_0, B_1, B_2 a single constant b enters the solution; its value is determined from the conditions of momentum flux conservation.

Equation (1.28) will under these conditions take the form

$$F'' + \frac{4b^2+1}{4(1+\omega)^2} F = 0. \tag{3.4}$$

Its solution must be sought in the form of

$$F = (1+\omega)^n, \tag{3.5}$$

to which the two independent particular solutions

$$F_1 = (1+\omega)^{1/2} \cos \Omega, \quad F_2 = (1+\omega)^{1/2} \sin \Omega,$$

correspond where $\Omega = b \ln(1+\omega)$.

A general solution to the dynamic problem, the definite function $f_1(\omega)$, is obtained in the form

$$f_1(\omega) = -2(1-\omega^2) \frac{F_1' + CF_2'}{F_1 + CF_2} = (1-\omega) \left\{-1 + 2b \frac{\sin \Omega - C \cos \Omega}{\cos \Omega + C \sin \Omega}\right\}. \tag{3.6}$$

The value of the arbitrary constant C can be found from the boundary conditions: $f_1(\cos \alpha) = f_1(\omega_\alpha) = 0$, i.e., $v_\theta = 0$ at the surface of the cone whose vertex angle is equal to 2α. This yields

$$C = \frac{2b \operatorname{tg} \Omega_\alpha - 1}{\operatorname{tg} \Omega_\alpha + 2b}. \tag{3.7}$$

Taking this expression into account we obtain finally

$$f_1(\omega) = N_\alpha (1-\omega) \frac{1 - \operatorname{tg} \Omega_\alpha \operatorname{ctg} \Omega}{M_\alpha \operatorname{ctg} \Omega - 1}, \tag{3.8}$$

with the additional definitions:

$$N_\alpha = \frac{4b^2+1}{1-2b \operatorname{tg} \Omega_\alpha}, \quad M_\alpha = \frac{2b + \operatorname{tg} \Omega_\alpha}{1 - 2b \operatorname{tg} \Omega_\alpha}. \tag{3.9}$$

As regards the constant b it must remain below a maximum value

which is determined from the transcendental equation

$$\frac{\{2b+\operatorname{tg}[b\ln(1+\cos\alpha)]\}\operatorname{ctg}(b\ln 2)}{1-2b\operatorname{tg}[b\ln(1+\cos\alpha)]}=1. \qquad (3.10)$$

In this equation, which results from the requirement of regularity of the velocity components throughout the field of the flow, the constant b is an implicit function of the characteristic Reynolds number of the problem, $\mathrm{Re}=\frac{2}{\sqrt{\pi}}\sqrt{\frac{J_x}{\rho\nu^2}}$ (J_x is the total momentum flux of the jet). As Re increases the value of the constant b also grows which corresponds to a transition to a "strong" jet.

Expression (3.8) obtained for the function $f_1(\omega)$ determines the velocity components and the pressure (when we take Eq. (1.26) for the functions $f_1(\omega)$ and $h_2(\omega)$ into account).

Let us also determine the function $\tau_1(\omega)$ entering the expression for the temperature. For this purpose we shall make use of Eq. (1.30) which links the sought function $F = F_1 + CF_2$ obtained above. After a few transformations we obtain

$$\tau_1(\omega)=B(1+\omega)^{-\frac{\mathrm{Pr}\,N_a}{M_a^2+1}(1+M_a\operatorname{tg}\Omega_a)}\times$$
$$\times\{M_a\cos\Omega-\sin\Omega\}^{-\frac{\mathrm{Pr}\,N_a}{b(M_a^2+1)}(M_a-\operatorname{tg}\Omega_a)}. \qquad (3.11)$$

The value of the constant B is determined from the condition of conservation of excessive heat content flux (under the assumption of an adiabatic insulation of the nonheat-conducting lateral surface of the cone):

$$Q=2\pi\lambda\int_{\omega_a}^{1}(1-\mathrm{Pr}\,f_1)\,\tau_1\,d\omega. \qquad (3.12)$$

The final expressions for the velocity components, the pressure and temperature are written in terms of the initial (spherical) coordinates:

$$
\left.
\begin{aligned}
v_R &= -\frac{\nu}{R}\frac{N_a}{M_a\cos\Omega-\sin\Omega}\left\{\operatorname{tg}\Omega_a\cos\Omega-\sin\Omega+\right.\\
&\qquad\left.+\,b\,\frac{1-\cos\theta}{1+\cos\theta}\,\frac{M_a-\operatorname{tg}\Omega_a}{M_a\cos\Omega-\sin\Omega}\right\},\\
v_\theta &= -\frac{\nu}{R}N_a\frac{1-\cos\theta}{\sin\theta}\,\frac{1-\operatorname{tg}\Omega_a\operatorname{ctg}\Omega}{M_a\operatorname{ctg}\Omega-1},\\
\frac{p-p_\infty}{\rho} &= -\frac{v_\theta^2}{2}+\nu\frac{v_R}{R}+\nu^2\frac{4b^2+1}{2R^2},\\
T-T_\infty &= \frac{B}{R}(1+\cos\theta)^{-\frac{\Pr N_a(1+M_a\operatorname{tg}\Omega_a)}{M_a^2+1}}\times\\
&\qquad\times\{M_a\cos\Omega-\sin\Omega\}^{-\frac{\Pr N_a(M_a-\operatorname{tg}\Omega_a)}{b(M_a^2+1)}}.
\end{aligned}
\right\}
\qquad (3.13)
$$

We also want to give the streamline equation which corresponds to the usual formula of the stream function of an axisymmetric motion:

$$\psi = \nu R f_1(\theta).$$

This equation reads

$$R=\frac{\text{const}}{f_1(\theta)}=\text{const}\,\frac{M_a-\operatorname{tg}\Omega}{(1-\cos\theta)(\operatorname{tg}\Omega-\operatorname{tg}\Omega_a)}. \qquad (3.14)$$

Figure 3.11 shows by way of example the stream lines for a motion corresponding to a value of the parameter $\alpha = \pi/3$. As we see from the figure, under the conditions chosen the flow pattern in the paraxial region maintains the characteristics form of a "strong" jet, in spite of the limitations imposed by the presence of the walls. This property is conserved within a comparatively wide range of values of the angle α.

The problem considered is interesting as an idealized scheme of expansion of a jet in a limited space. It is in particular characterized by counterflows of the fluid in the space between jet and walls which are typical of such flows. For the jet of a source as considered here it is obvious that the total flow rate of the fluid through an arbitrary orthogonal cone of spherical surface will be equal to zero.

Among the great number of possible values of the vertex angle 2α of the cone we select one, $\alpha = \pi/2$, which corresponds to the problem of

a jet efflux from an opening in a plane wall. This flow, a particular case of the problem under consideration, has already been studied by H. Squire [303].

With $\alpha = \pi/2$, i.e., a jet perpendicular to the plane wall, we obtain from Eq. (3.13)

$$\left.\begin{aligned} v_R &= \frac{\nu}{R}\frac{4b^2+1}{2b\cos\Omega-\sin\Omega}\Big\{\sin\Omega - \\ &\quad -2b^2\frac{1-\cos\theta}{1+\cos\theta}\frac{1}{2b\cos\Omega-\sin\Omega}\Big\}, \\ v_\theta &= -\frac{\nu}{R}\frac{1-\cos\theta}{\sin\theta}\frac{4b^2+1}{2b\operatorname{ctg}\Omega-1}, \\ \frac{p-p_\infty}{\rho} &= -\frac{v_\theta^2}{2}+\nu\frac{v_R}{R}+\nu^2\frac{4b^2+1}{2R^2}, \\ T-T_\infty &= \frac{B}{R}(1+\cos\theta)^{-Pr}\{2b\cos\Omega - \\ &\quad -\sin\Omega\}^{-2\,Pr}. \end{aligned}\right\} \qquad (3.15)$$

The diagrams of Fig. 3.12 illustrate this flow. For a "strong" jet the influence of the walls on the flow in the paraxial region is extremely weak. When calculating the motion within the framework of the boundary layer theory this fact enables us to ignore the walls' influence on the laws governing the expansion of a jet.

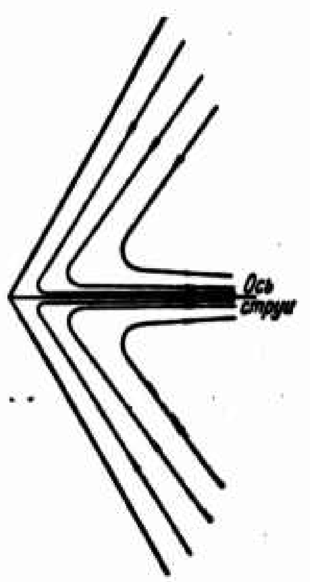

Fig. 3.11. Streamlines for jet in cone ($\alpha = \pi/3$). 1) Jet axis.

The boundary condition of the theory of a perfect fluid ($v_\theta = 0$, $v_R \neq 0$ at the cone surface) accepted in this section instead of the physically correct condition describing the adhension of the visous fluid, in the case of a "strong" jet, virtually does not cause a distortion in the paraxial flow, not even at a given distance from the axis but sufficiently far away from the walls, since the boundary layer thickness at the latter is relatively small. For more precise qualitative details

reference must be made to N.A. Slezkin's paper [172].

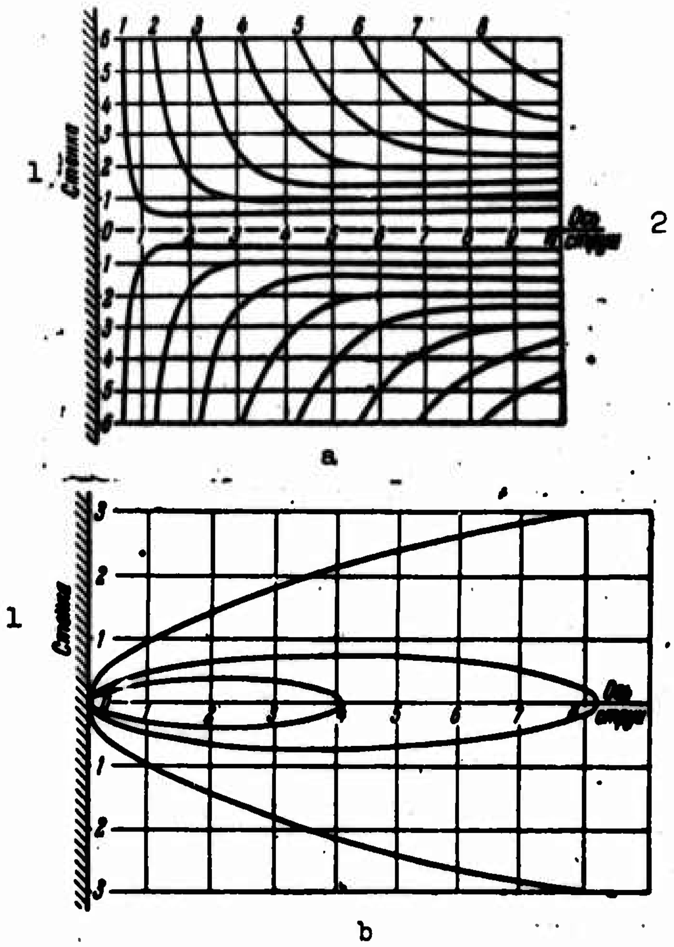

Fig. 3.12. Streamlines a) And isothermal lines b) in a jet issued from an opening in a plane wall]303]. 1) wall; 2) axis of jet.

Let us finally mention that the solution obtained for a jet issued from a plane wall (just as for the cone) can easily be completed in order also to comprise the case of a twisted jet. The solution of this problem will be considered in detail in the following section for a fan-type twisted jet. Here, on the basis of the similarity of these solutions, we only give the final expression for the rotational velocity component for the efflux of a jet out of a wall, without deriving it:

$$v_\varphi = \frac{C_4}{R^2}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\,\frac{(1+\cos\theta)^b}{[b+1+(b-1)(1+\cos\theta)^b]^2}. \qquad (3.15a)$$

The constant $C_4 = 4C_3b^2$ can be determined when the angular momentum

flux M_x relative to the longitudinal axis is given.

This expression is identical to the analogous one for the type jet (see below) as in both cases the expressions of the function $f_1(\omega)$ and $\Phi_2(\omega)$ are the same and there does not exist any twist on the axis: $\frac{\Phi_2}{\sqrt{1-\omega^2}} = 0$ with $\omega = 1$. Only v_φ in cylindrical coordinates will be different.

3.3. THE TWISTED FAN-TYPE JET

Just as in the above case we consider the values of the constants B_0, B_1 and B_2 of Eqs. (1.28) connected with the equation $B_0 = B_2 = -\frac{1}{2} B_1$. The numerical values of the constants is assumed equal to $B_0 = 1 - b^2/2$. In this case Eq. (1.28) and its solution under the correponding boundary conditions will describe a flow produced by an annular source*. As already mentioned in the Introduction we agreed in calling this form of jet motion fan-type or radial jet (see schematic representation of Fig. 0.2).

With the values accepted for the constants B_0, B_1 and B_2 Eq. (1.28) can be written in the form

$$F'' + \frac{1-b^2}{4(1+\omega)^2} F = 0. \tag{3.16}$$

Its particular solutions have the form

$$F_1 = (1+\omega)^{\frac{1}{2}(1+b)}, \quad F_2 = (1+\omega)^{\frac{1}{2}(1-b)},$$

and hence we obtain

$$f_1(\omega) = -2(1-\omega^2)\frac{F_1' + CF_2'}{F_1 + CF_2} = (1-\omega)\left\{-1 + b\frac{C-(1+\omega)^b}{C+(1+\omega)^b}\right\}. \tag{3.17}$$

The boundary conditions for the fan-type jet are the following:

$$f_1 = 0 \quad \text{when} \quad \omega = 0, \quad f_1 = 0 \quad \text{when} \quad \omega = 1, \tag{3.18}$$

they permit the determination of the arbitrary constant C

$$C = \frac{b+1}{b-1}, \tag{3.19}$$

whence

$$f_1(\omega)=(1-\omega)\left\{-1+b\frac{\frac{b+1}{b-1}-(1+\omega)^b}{\frac{b+1}{b-1}+(1+\omega)^b}\right\}. \qquad (3.20)$$

The constant b, as in other cases, is determined by the integral condition of momentum flux conservation (for the semispace)

$$J_x=2\pi R^2\int_0^{+1}\left(p+\rho v_R^2-2\mu\frac{\partial v_R}{\partial R}\right)\sqrt{1-\omega^2}\,d\omega, \qquad (3.21)$$

which results in an implicit connection of b and the Reynolds number:

$$\mathrm{Re}\sim\sqrt{\frac{J_x}{\rho v^2}}.$$

Here too a "strong" jet corresponds to large values of Re.

Let us now turn to the calculation of the rotational velocity component v_φ in the same self-similar approximation.

The function $\Phi_2(\omega)$ entering expression $v_\varphi=\frac{v}{r^2}\frac{\Phi_2(\omega)}{\sqrt{1-\omega^2}}$, is determined by the differential equation

$$\Phi_2''-\frac{f_1}{1-\omega^2}\Phi_2'+\frac{2-f_1'}{1-\omega^2}\Phi_2=0. \qquad (3.22)$$

The boundary conditions, taking the symmetry with respect to the plane $\theta=\frac{\pi}{2}$ $(\omega=0)$ into account, can be written in the form

$$\Phi_2'(0)=0,\quad \lim_{\omega\to 1}\left(\frac{\Phi_2}{\sqrt{1-\omega^2}}\right)=0. \qquad (3.23)$$

Integrating Eq. (3.22), preliminarily rewritten in the form

$$[(1-\omega^2)\Phi_2']'+2(\omega\Phi_2)'-(f_1\Phi_2)'=0,$$

and taking the boundary conditions into account, we obtain

$$\Phi_2(\omega)=C_3(1-\omega^2)\exp\left(\int_0^\omega\frac{f_1\,d\omega}{1-\omega^2}\right).$$

Using expression $f_1(\omega)$ derived above we obtain finally

$$\Phi_2(\omega)=C_4\frac{(1-\omega)(1+\omega)^b}{[b+1+(b-1)(1+\omega)^b]^2}, \qquad (3.24)$$

where $C_4=4C_3b^2$. The constant C_4 is determined by the total flux of angular momentum relative to the axis of symmetry of the jet, through a semisphere of arbitrary radius centered at the source of the jet:

$$M_x = 2\pi\rho\nu^2\int_0^1 (3-f_1)\,\Phi_2\,d\omega. \tag{3.25}$$

In order to solve the thermal problem under the condition that the temperature values in the upper and the lower semispaces must be equal we again use Eq. (1.30).

After simple transformations we obtain

$$\tau_1(\omega) = B(1+\omega)^{(b-1)\Pr}\left[\frac{b+1}{b-1}+(1+\omega)^b\right]^{-2\Pr}. \tag{3.26}$$

The value of the constant of integration B is determined as before, from the integral condition of conservation of excessive heat content flux:

$$Q = 2\pi\lambda\int_0^1 (1-\Pr f_1)\,\tau_1\,d\omega. \tag{3.37}$$

The final expressions in spherical coordinates for the velocity components, the pressure and temperature have the form

$$\left.\begin{aligned}
v_R &= -\frac{\nu}{R}\left\{1 - b\,\frac{\frac{b+1}{b-1}-(1+\cos\theta)^b}{\frac{b+1}{b-1}+(1+\cos\theta)^b} - {}\right.\\
&\qquad\left. -2b^2\,\frac{b+1}{b-1}\,\frac{1-\cos\theta}{1+\cos\theta}\,\frac{(1+\cos\theta)^b}{\left[\frac{b+1}{b-1}+(1+\cos\theta)^b\right]^2}\right\},\\
v_\theta &= \frac{\nu}{R}\,\frac{1-\cos\theta}{\sin\theta}\left\{1 - b\,\frac{\frac{b+1}{b-1}-(1+\cos\theta)^b}{\frac{b+1}{b-1}+(1+\cos\theta)^b}\right\},\\
v_\varphi &= \frac{\nu C_4}{R^2}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\,\frac{(1+\cos\theta)^b}{[b+1+(b-1)(1+\cos\theta)^b]^2},\\
\frac{p-p_\infty}{\rho} &= -\frac{v_\theta^2}{2} + \nu\frac{v_R}{R} - \frac{\nu^2}{2R^2}(b^2-1),\\
T-T_\infty &= \frac{B}{R}(1+\cos\theta)^{\Pr\,(b-1)}\left[\frac{b+1}{b-1}+(1+\cos\theta)^b\right]^{-2\Pr}.
\end{aligned}\right\} \tag{3.28}$$

As to the qualitative aspects the graphs of streamlines and isothermal lines corresponding to the solution obtained displays the same characteristic properties of a "strong" jet (with the appropriate values of the problem's parameters) as in L.D. Landau's problem considered in detail. Not that the solution obtained to the dynamical problem is, in particular, free from the deficiencies displayed by H. Squire's

paper [171]. In this paper the streamlines must be constructed under the assumption that they emerge from fictitious sources positioned on the axis of symmetry of the jet. The author of paper [171] points out that these sources are unreal but their introduction distorts the jet flow only little.

In this problem, just as in all the problems considered above, the thermal problem is solved under the presupposition of similar boundary conditions for velocity and temperature. In contrast to this, the fan-type jet enables us to investigate the thermal problem for the case of a temperature asymmetry in the boundary conditions and, consequently, lost similarity of the boundary conditions for velocity and temperature.

Note that the temperature values of the nonperturbed fluid differ in the upper and lower spaces, that is, on either side of the plane of symmetry of the fan-type jet. Assume that with $R \to \infty$ and $\omega = 1$ the temperature $T = T_1$ whereas with $R \to \infty$ but $\omega = -1$, $T = T_2$. In this case, in order to solve the thermal problem, using the same results as obtained when solving the dynamical problem, we must, instead of using Eq. (1.29), return to the basic equation of heat propagation, i.e., Eq. (1.5).

From physical considerations analogous to those of paper [56] we are led to

$$\frac{T - T_2}{T_1 - T_2} = \tau(\theta), \qquad (3.29)$$

i.e., the temperature at any point of the flow produced by the fan-type source jet is assumed independent of the distance to the source. This is obviously the only condition which permits a self-cimilar solution with the temperature boundary conditions chosen. In agreement with (e.29), Eq. (1.5) can be rewritten in the form

$$\frac{Rv_\theta}{a}\frac{d\tau}{d\theta} = \frac{d^2\tau}{d\theta^2} + \frac{d\tau}{d\theta}\operatorname{ctg}\theta. \qquad (3.30)$$

Introducing the variable $\omega = \cos\theta$ and substituting the value of v_θ, we obtain for $\tau(\omega)$ an ordinary linear differential equation of second order:

$$[(1-\omega^2)\tau'(\omega)]' = \Pr f_1(\omega)\tau'(\omega) \tag{3.31}$$

with the boundary conditions

$$\left.\begin{aligned} \tau = 1 \quad &\text{при} \quad \omega = +1, \\ \tau = 0 \quad &\text{при} \quad \omega = -1. \end{aligned}\right\} \tag{3.32}$$

When we replace in Eq. (3.31) the function $f_1(\omega)$ by the function $F(\omega)$, which is connected with $f_1(\omega)$ by the relation (1.27), we obtain

$$\frac{d\ln[(1-\omega^2)\tau']}{d\ln F} = -2\Pr. \tag{3.33}$$

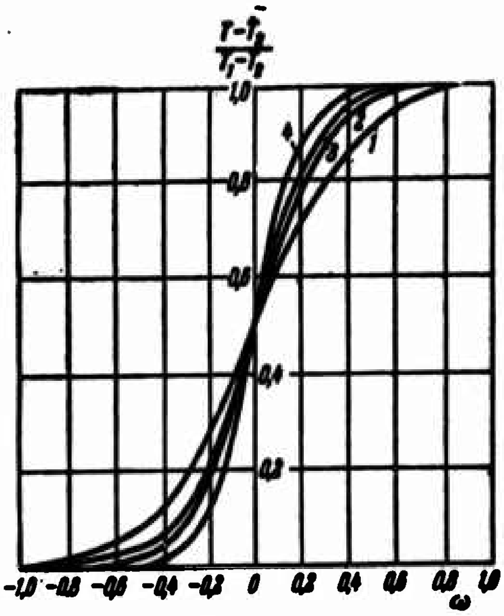

Fig. 3.13. Temperature distribution in a fan-type jet. $b = 4$; 1. Pr = 0.5; 2. Pr = 0.75; 3. Pr = 1.0; 4. Pr = 2.0.

Integrating twice and taking the boundary conditions (3.32) into account, we obtain

$$\tau(\omega) = \int_{-1}^{\omega} \frac{d\omega}{(1-\omega^2)F^{2\Pr}} \left[\int_{-1}^{+1} \frac{d\omega}{(1-\omega^2)F^{2\Pr}}\right]^{-1}. \tag{3.34}$$

When we take the expression for the function $F(\omega)$ from H. Squire's paper [171]

$$F=(1-\omega^2)^{1/2}\left[\left(\frac{1-\omega}{1+\omega}\right)^{b/2}+\left(\frac{1-\omega}{1+\omega}\right)^{-b/2}\right],$$

we can write our expression for the temperature in the final form

$$\tau(\omega)=\int_{-1}^{\omega}\frac{d\omega}{(1-\omega^2)^{Pr+1}\left[\left(\frac{1-\omega}{1+\omega}\right)^{b/2}+\left(\frac{1-\omega}{1+\omega}\right)^{-b/2}\right]^{2Pr}}\times$$
$$\times\left[\int_{-1}^{+1}\frac{d\omega}{(1-\omega^2)^{Pr+1}\left[\left(\frac{1-\omega}{1+\omega}\right)^{b/2}+\left(\frac{1-\omega}{1+\omega}\right)^{-b/2}\right]^{2Pr}}\right]^{-1}. \quad (3.35)$$

The corresponding temperature distribution for several values of Prandtl's number Pr is shown in Fig. 3.13.

3.4. TRANSITION TO THE BOUNDARY LAYER

Analyzing the solutions obtained above, one often considered the laws of a flow, corresponding to one of the limiting regions, namely the "strong" jet. By way of example L.D. Landau's basic problem on the axisymmetric source jet, we shall now show that the relations obtained in the limiting transition (with $Re \to \infty$), transform the solutions found in the general case into solutions to the same problem within the framework of the boundary layer theory. This result is of course not unexpected but it is of interest as it shows in particular that the solutions of the Navier-Stokes equations contain a series of solutions obtained independently at a different time.

For the sake of convenience we now turn to the system of equations (3.1) written in cylindrical coordinates. In this system the velocity component, pressure and temperature at the axis of the jet depend on the coordinate x and the parameter L which are unambiguous functions of Reynolds' number. The relative distributions of the velocity components, the pressure and the temperature in the cross sections of the jet are in their turn dependent on the nondimensional coordinate $\eta = r/x$ and the parameter L. In addition to its dependence on other parameters the temperature distribution also depends on Prandtl's number.

Let in all equations of the system (3.1) L tend subsequently to

unity (Re $\to \infty$). We consider preliminarily Eq. (2.10), which links the momentum J_x of the jet with the parameter L:

$$J_x = \frac{16\pi\rho\nu^2}{L}\left\{1 + \frac{4L^2}{3(1-L^2)} - \frac{1}{2L}\ln\frac{1+L}{1-L}\right\}.$$

With $L \to 1 (J_x \to \infty)$ we shall have

$$L = 1 - \frac{\alpha^2}{2}, \qquad \alpha^2 = \frac{64\pi\rho\nu^2}{3J_x}. \tag{3.36}$$

The nondimensional parameter α is obviously inversely proportional to Reynolds' number Re.

Using Eq. (3.36) we can write the equations of the system (3.1) in a form which corresponds to the limiting transition $L \to 1$ (Re $\to \infty$). In all expressions we restrict ourselves to terms which are small in second order with respect to the quantities α and η. Note that we are here concerned with one and not with two independent limitations as we learn from dimension considerations that for a "strong" jet the quantities α and η are of the same order of magnitude.

To illustrate this we shall varry out in detail the transformation of the expression for one of the velocity components, v_x.
In a general case

$$\frac{x}{2\nu} v_x = \frac{L}{\sqrt{1+\eta^2}}\left\{1 + \frac{1-L^2}{(\sqrt{1+\eta^2}-L)^2}\right\}.$$

With the substitution $L = 1-\alpha^2/2$ we have

$$\frac{x}{2\nu} v_x \approx \frac{1-\frac{\alpha^2}{2}}{1+\frac{1}{2}\eta^2}\left\{1 + \frac{1-\left(1-\frac{\alpha^2}{2}\right)^2}{\left[1+\frac{\eta^2}{2}-\left(1-\frac{\alpha^2}{2}\right)\right]^2}\right\}$$

or

$$\frac{x}{2\nu} v_x \approx \left(1-\frac{\alpha^2}{2}\right)\left(1-\frac{1}{2}\eta^2\right)\left[1 + \frac{4\alpha^2}{(\alpha^2+\eta^2)^2}\right],$$

or, finally,

$$\frac{x}{2\nu} v_x \approx \frac{4}{\alpha^2}\frac{1}{\left(1+\frac{\eta^2}{\alpha^2}\right)^2}. \tag{3.37}$$

Recall that $\alpha^2 = \frac{64\pi\rho\nu^2}{3J_x}$, and when we introduce the designation

$$\varphi = \frac{r}{x}\sqrt{\frac{3J_x}{64\pi\rho\nu^2}}. \tag{3.38}$$

we finally obtain

$$v_x \approx \frac{3J_x}{8\pi\rho\nu} \frac{1}{x} \frac{1}{\left(1+\frac{1}{8}\varphi^2\right)^2}. \tag{3.39}$$

Without giving the analogous transformations, the other equations of system (3.1) can be written in the transformed forms:

$$v_r \approx \frac{1}{2}\sqrt{\frac{3J_x}{8\pi\rho}} \frac{1}{x} \frac{\varphi\left(1-\frac{1}{8}\varphi^2\right)}{\left(1+\frac{1}{8}\varphi^2\right)^2}, \tag{3.40}$$

$$v_\varphi \approx \frac{3M_x}{32\pi\rho\nu^2} \sqrt{\frac{3J_x}{8\pi\rho}} \frac{1}{x^2} \frac{\varphi}{\left(1+\frac{1}{8}\varphi^2\right)^2}, \tag{3.41}$$

$$\frac{p-p_\infty}{\rho} \approx \frac{8\nu^2}{a^2} \frac{1}{x^2} \frac{1-\frac{1}{8}\varphi^2}{\left(1+\frac{1}{8}\varphi^2\right)^2}, \tag{3.42}$$

$$T-T_\infty \approx \frac{Q\,(2Pr+1)}{8\pi\rho\nu c_p} \frac{1}{x} \frac{1}{\left(1+\frac{1}{8}\varphi^2\right)^{2Pr}}. \tag{3.43}$$

The last expression for the temperature was derived by means of a limiting expression for the constant B from Eq. (2.12)

$$B \approx \frac{Q}{\pi\lambda} \frac{2Pr+1}{Pr} 2^{-Pr-3} a^{4Pr}.$$

Let us briefly discuss the results obtained.

First of all we want to represent the distributions of the velocity components, the pressure and the temperature under the limiting conditions chosen in equivalent forms:

$$\left.\begin{aligned}
\frac{v_x}{v_{xm}} &= \frac{1}{\left(1+\frac{1}{8}\varphi^2\right)^2},\\
\frac{v_r}{v_{xm}} &= \frac{1}{2}\sqrt{\frac{8\pi\rho\nu^2}{3J_x}} \frac{\varphi\left(1-\frac{1}{8}\varphi^2\right)}{\left(1+\frac{1}{8}\varphi^2\right)^2},\\
\frac{v_\varphi}{v_{xm}} &= \frac{M_x}{4\nu J_x} \frac{1}{x} \sqrt{\frac{3J_x}{8\pi\rho}} \frac{\varphi}{\left(1+\frac{1}{8}\varphi^2\right)^2},\\
\frac{p_m-p_\infty}{\left(\rho\frac{v_x^2}{2}\right)_m} &= \frac{a^2}{4} \approx 0,\\
\frac{T-T_\infty}{T_m-T_\infty} &= \frac{1}{\left(1+\frac{1}{8}\varphi^2\right)^{2Pr}}.
\end{aligned}\right\} \tag{3.44}$$

where the quantities v_{xm} and $T_m - T_\infty = \Delta T_m$, are taken as the velocity and temperature scale units given by

$$v_{xm} = \frac{3J_x}{8\pi\rho\nu}\frac{1}{x}, \qquad \Delta T_m = \frac{(2Pr+1)Q}{8\pi\rho\nu c_p}\frac{1}{x}. \qquad (3.45)$$

As we see from the above formulas, the transverse velocity components and temperature distributions in a "strong" jet are universal functions (each quantity being referred to their scale units, i.e., the peak values). The velocity components v_x and v_R and also the surplus temperature $T - T_\infty$ decrease with the distance according to an $1/x$-law, whereas $v_\varphi \sim 1/x^2$. The relative amount of "twisting", v_φ/v_x will therefore drop as $1/x$.

As regards the pressure, in our approximation $p_m - p_\infty \sim 1/a_1 \sim J_x$ the surplus pressure on the jet axis (and thus also at any arbitrary point) will grow proportionally to the momentum of the jet. The relative quantity of the order of $\frac{p_m - p_\infty}{(\rho v_x^2/2)_m}$, will be a quantity of the order of $a^2 \ll 1$. as we can see from Eq. (3.44). In the approximation considered the flow may be considered isobaric.

Not also that from the fact that the quantities α and η are of the same order of magnitude it results that the effective region of the flow occupied by the jet must be limited by a cone of a relatively small vertex angle $\left(\frac{r}{x} = \operatorname{tg}\alpha_0 \sim \alpha_0 \sim \alpha\right)$.

In this case the ratio v_r/v_x of the velocity components will be a small quantity of the order of $\alpha \sim 1/\mathrm{Re}$; this is typical for a "strong" jet. Analogously, the ratio v_φ/v_x of the components will also be a small quantity of the order of $\alpha M_x/x$ as we are concerned with slight "twist".

These qualitative results obtained for a "strong" jet, such as the limitedness of the perturbed region, the pressure distribution, the the smallness of the radial velocity component compared to the axial

one, kthe rapid change of the velocity and the like across the jet compared to the variations along it, are well-know characteristic features of the flow in a boundary layer.

More than that, Expressions (3.40)-(3.45) derived above are as accurate as the results of a direct solution of the problem of the expansion of an axisymmetric twisted laminar source jet, obtained by L.G. Loytsyanskiy [131] by intgrating the boundary layer equations (see Part II).

Our limiting transition ($L \to 1$) corresponds to a first self-similar approximation of the solution to L.D. Landau's problem. The final result obtained is a first self-similar solution of the problem of an axisymmetric source-jet within the framework of the boundary-layer theory. The transition $L \to 1$ made to obtain the second and further approximations in solutions of Landau's problem would of course also yield the second and further approximations in the boundary layer theory.

Let us give for comparison, also in this first approximation, the formulas obtained by another limiting transition ($L \to 0$) for the case of a "weak" jet. As indicated above this corresponds to a momentum of $J_x = 16\pi\rho\nu^2 L$, of the jet, that is, to small Reynolds numbers

$$\mathrm{Re} = \frac{2}{\sqrt{\pi}} \sqrt{\frac{J_x}{\rho\nu^2}} \to 8\sqrt{L}.$$

Without giving details of the transformations we write the final results:

$$\left.\begin{aligned}
&\frac{v_x}{v_{xm}} \approx \frac{2+\eta^2}{2(1+\eta^2)^{3/2}}, \\
&\frac{v_r}{v_{xm}} \approx \frac{\eta}{2(1+\eta^2)^{3/2}}, \qquad v_{xm} = \frac{J_x}{4\pi\rho\nu}\frac{1}{x}, \\
&\frac{v_\varphi}{v_{xm}} \approx \frac{3M_x}{2xJ_x}\frac{\eta}{(1+\eta^2)^{3/2}}, \\
&\frac{p-p_\infty}{\left(\rho\frac{v_x^2}{2}\right)_m} \approx \frac{8\pi\rho\nu^2}{J_x}\frac{1}{(1+\eta^2)^{3/2}}, \qquad \Delta T_m = \frac{Q}{4\pi\lambda}\left(\frac{J_x}{16\pi\rho\nu^2}\right)^{-2\mathrm{Pr}}\frac{1}{x}. \\
&\frac{\Delta T}{\Delta T_m} \approx \frac{1}{\sqrt{1+\eta^2}}.
\end{aligned}\right\} \qquad (3.46)$$

As we see from these formulas, the velocity components v_x and v_r in a "weak" jet, in contrast to the "strong" jet, are quantities of the same order of magnitude. As regards the pressure, $\Delta p/(\rho v_x^2/2)_m$ is of the order of Re^2; in other words, the pressure drop plays a noticeable part in the motion of the fluid compared to the kinetic hear $\rho v_x^2/2$.

Thus the motion of a viscous fluid in the limiting case of a "weak" jet rather reminds us of a source in a perfect fluid than of a jet flow characteristics of problems of the boundary layer theory.

The jet flow properties established with the help of the example of an axisymmetric jet are dur to the fact that as $Re \to \infty$ the influence of the viscosity seems to become localized to a samll zone, the boundary layer, which is common for flows of viscous fluids. In the general boundary layer theory considered in the following in connection with jet flows, the limiting transition $Re \to \infty$ will therefore be carried out in the Navier-Stokes equations themselves.

REFERENCES

116, 122, 171, 172, 173, 174, 202, 203, 221, 302, 303.

Part Two

LAMINAR JETS OF LIQUIDS AND GASES

In this part we shall consider a series of problems on the expansion of a jet of liquid or gas which are solved within the framework of the laminar boundary layer theory.

As shown in the preceding part these solutions (for an incompressible fluid) correspond to the limiting transition in solutions obtained from the Navier-Stokes equations for a "strong" jet. Unlike this case, the analogous problems (e.g., on the expansion of an axisymmetric jet) in the second part are considered on the basis of an analogous limiting transition carried out in an earlier stage, that is, in the initial differential equations, which is thus general for the various problems.

For simplicity this transition (derivation of the boundary layer equations) is carried out for motions of incompressible fluids.

This derivation and also a brief review of the various methods of integration of the laminar boundary layer equations applied in the theory of jet motion of liquids and gases, will be given before we turn to the analyses of concrete jet problems. The latter are considered separately for the cases of free and semi-limited jet flows of liquids and gases, taking the specific features into account (the form of the flow, boundary conditions, etc.). The consideration is limited to self-similar motions, i.e., an investigation of source jets.

For these jets detailed information, containing the initial equations, the boundary conditions, the formulas for the transformation to a self-similar form, the final solutions and the integral characteristics are for convenience compiled in the tabke of the fundamental results. For a jet of finite dimensions a solution is only given within

the framework of the method of small perturbations. In all cases preference is given to the method of the asymptotic boundary layer principally used in this book; for comparison, however, we also give in a brief form the solutions obtained by the method of integral relations for a free boundary layer of finite thickness.

The results obtained in this part have both an immediate significance (for the theory of the laminary boundary layer) and an indirect one, as a model of a turbulent jet flow.

Chapter 4

FREE JETS OF AN INCOMPRESSIBLE FLUID

4.1. THE BOUNDARY LAYER EQUATION

It is well-known that the number of problems on the motion of an incompressible fluid that can be solved on the basis of the exact Navier-Stokes equation is extremely limited. Even a solution of the problem on the expansion of a plane source jet (analogous to L.D. Landau's problem on the axisymmetric jet) applicable to jet flows has not been obtained so far*. In connection with this, besides the numerical solutions of concrete problems in the theory of laminar jets, the general method of solution based on the widespread boundary layer theory is of great importance.

The methods of the boundary layer theory are now applied in virtually all fields of hydrodynamics and gasdynamics; there is very much literature devoted to it [130, 135, 155, 174, 212, etc.]. To recall them we shall therefore consider as briefly as possible by way of example of a steady laminar flow of incompressible fluid in a boundary layer (plane and axisymmetric) the transition from the Navier-Stokes equations to Prandtl's boundary layer equations.

The ideas which is based have already been used above when we explained the solurion to L.D. Landau's problem on the axisymmtric jet, in the "strong"jets. As shown above in detail, the latter correspone to the limiting transition $Re \to \infty$ in the solutions of the Navier-Stokes equations. In this transition the flow displays a peculiar anisotropy: the characteristic dimensions and velocity components in a direction

of the initial momentum become very small compared to the analogous quantities in the direction of the momentum. In a short form these relationships may be represented by the following inequalities **

$$l_x \gg l_y, \quad u \sim w \gg v \qquad (u = v_x, \quad v = v_y, \quad w = v_\varphi)$$

and consequently,

$$\frac{\partial}{\partial x} \ll \frac{\partial}{\partial y}.$$

To estimate the individual terms entering the Navier-Stokes equations it must be assumed that the quantities l_x, u and w are of the order of unity, l_y and v being of the order of a small quantity $\delta \ll 1$ (correspondingly the orders of magnitude of the derivatives $\partial/\partial x$, $\partial^2/\partial x^2$ will be unity while the derivatives $\partial/\partial y, \partial^2/\partial y^2$ will be of the order of δ^{-1}, δ^{-2}). Starting from this and from the tendency of obtaining an approximation in which the viscosity factor μ is conserved in the equations, we must assume μ of the order of δ^2. This constitutes in principle the difference between the transition to the boundary layer equations and the transition to Euler's equations of a perfect fluid in which one must set $\mu \equiv 0$.

To avoid the difficulty caused by this way of estimating a dimensional physical constant ($\mu \sim \delta^2$) the transformation to the boundary layer equation can be carried out with the nondimensional form of the Navier-Stokes equations, introducing in them different scale units for the longitudinal and transverse coordinates and velocity components. In this case the boundary layer equations are obtained by means of a limiting transition, letting tend the Reynolds number to infinity.

Taking this conclusion into account we write the Navier-Stokes and energy equations in a form which is suitable for both the axisymmetric (in the case of a nonvanishing peripheral velocity component) and the plane-parallel motions:

$$
\begin{aligned}
&u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left\{\frac{1}{y^k}\frac{\partial}{\partial y}\left(y^k\frac{\partial u}{\partial y}\right)+\frac{\partial^2 u}{\partial x^2}\right\},\\
&u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}-\frac{\tilde{w}^2}{y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left\{\frac{1}{y^k}\frac{\partial}{\partial y}\left(y^k\frac{\partial v}{\partial y}\right)-k\frac{v}{y^2}+\frac{\partial^2 v}{\partial x^2}\right\},\\
&u\frac{\partial \tilde{w}}{\partial x}+v\frac{\partial \tilde{w}}{\partial y}+\frac{v\tilde{w}}{y}=\nu\left\{\frac{1}{y^k}\frac{\partial}{\partial y}\left(y^k\frac{\partial \tilde{w}}{\partial y}\right)+\frac{\partial^2 \tilde{w}}{\partial x^2}-\frac{\tilde{w}}{y^2}\right\}\ (\tilde{w}=kw),\\
&\frac{\partial}{\partial x}(y^k u)+\frac{\partial}{\partial y}(y^k v)=0,\\
&u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=a\left\{\frac{1}{y^k}\frac{\partial}{\partial y}\left(y^k\frac{\partial T}{\partial y}\right)+\frac{\partial^2 T}{\partial x^2}\right\}.
\end{aligned}
$$

(The axis O_x is oriented along the initial momentum of the jet, the axis Oy is perpendicular to Ox; u and v are the velocity components along Ox and Oy; w is the peripheral velocity; $k = 1$ for an axisymmetric motion, $k = 0$ for a plane-parallel one.)

Let us introduce the dimensionless quantities

$$
\bar{x}=\frac{x}{l_x},\ \bar{y}=\frac{y}{l_y},\ \bar{u}=\frac{u}{u_0},\ \bar{v}=\frac{v}{v_0},\ \bar{w}=\frac{w}{u_0},
$$
$$
\bar{p}=\frac{p-p_\infty}{\frac{1}{2}\rho u_0^2},\ \bar{T}=\frac{T}{T_\infty},\ \mathrm{Re}=\frac{u_0 l_x}{\nu},
$$

where l_x, l_y, u_0, v_0 are the scale units (the characteristic dimensions and velocities), Re is Reynolds' number, p_∞, T_∞ are pressure and temperature in the unperturbed fluid.

For simplicity we omit the bars on the dimensionless variables so that we can write the Navier-Stokes and energy equations for the considered case of a steady laminar flow of an incompressible fluid as follows:

$$
\left.
\begin{aligned}
&u\frac{\partial u}{\partial x}+\frac{l_x v_0}{l_y u_0}v\frac{\partial u}{\partial y}=-\frac{1}{2}\frac{\partial p}{\partial x}+\frac{1}{\mathrm{Re}}\left\{\frac{\partial^2 u}{\partial x^2}+\frac{l_x^2}{l_y^2}\cdot\frac{1}{y^k}\frac{\partial}{\partial y}\left(y^k\frac{\partial u}{\partial y}\right)\right\},\\
&u\frac{\partial v}{\partial x}+\frac{l_x v_0}{l_y u_0}v\frac{\partial v}{\partial y}-k\frac{l_x u_0}{l_y v_0}\cdot\frac{w^2}{y}=-\frac{l_x u_0}{l_y v_0}\cdot\frac{1}{2}\frac{\partial p}{\partial y}+\\
&\qquad+\frac{1}{\mathrm{Re}}\left\{\frac{\partial^2 v}{\partial x^2}+\frac{l_x^2}{l_y^2}\left[\frac{1}{y^k}\frac{\partial}{\partial y}\left(y^k\frac{\partial v}{\partial y}\right)-k\frac{v}{y^2}\right]\right\},\\
&u\frac{\partial \tilde{w}}{\partial x}+\frac{l_x v_0}{l_y u_0}\left[v\frac{\partial \tilde{w}}{\partial y}+\frac{v\tilde{w}}{y}\right]=\frac{1}{\mathrm{Re}}\left\{\frac{\partial^2 \tilde{w}}{\partial x^2}+\frac{l_x^2}{l_y^2}\left[\frac{1}{y^k}\frac{\partial}{\partial y}\left(y^k\frac{\partial \tilde{w}}{\partial y}\right)-\frac{\tilde{w}}{y^2}\right]\right\},\\
&\frac{\partial}{\partial x}(y^k u)+\frac{l_x v_0}{l_y u_0}\frac{\partial}{\partial y}(y^k v)=0,\\
&u\frac{\partial T}{\partial x}+\frac{l_x v_0}{l_y u_0}\cdot v\frac{\partial T}{\partial y}=\frac{a}{\nu}\cdot\frac{1}{\mathrm{Re}}\left\{\frac{\partial^2 T}{\partial x^2}+\frac{l_x^2}{l_y^2}\frac{1}{y^k}\frac{\partial}{\partial y}\left(y^k\frac{\partial T}{\partial y}\right)\right\}.
\end{aligned}
\right\}\quad (4.1)
$$

Taking into account that the scale units l_y and v_0 were chosen ar-

bitraily and considering the condition for the order-of-magnitude equality of the inertial and viscous terms as Re → ∞, we put

$$\frac{l_x}{l_y} = \frac{u_0}{v_0}, \quad \frac{l_x}{l_y} = \sqrt{\mathrm{Re}}.$$

Our system of equations will then take the form

$$\left.\begin{aligned}
&u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{2}\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial x^2} + \frac{1}{y^k}\frac{\partial}{\partial y}\left(y^k\frac{\partial u}{\partial y}\right),\\
&\frac{1}{\mathrm{Re}}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) - k\frac{w^2}{y} = -\frac{1}{2}\frac{\partial p}{\partial y} + \frac{1}{\mathrm{Re}^2}\frac{\partial^2 v}{\partial x^2} +\\
&\qquad\qquad + \frac{1}{\mathrm{Re}}\left\{\frac{1}{y^k}\frac{\partial}{\partial y}\left(y^k\frac{\partial v}{\partial y}\right) - k\frac{v}{y^2}\right\},\\
&u\frac{\partial \tilde{w}}{\partial x} + v\frac{\partial \tilde{w}}{\partial y} + \frac{v\tilde{w}}{y} = \frac{1}{\mathrm{Re}}\frac{\partial^2 \tilde{w}}{\partial x^2} + \frac{1}{y^k}\frac{\partial}{\partial y}\left(y^k\frac{\partial \tilde{w}}{\partial y}\right) - \frac{\tilde{w}}{y^2},\\
&\frac{\partial}{\partial x}(y^k u) + \frac{\partial}{\partial y}(y^k v) = 0,\\
&u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{\mathrm{Re}\cdot Pr}\cdot\frac{\partial^2 T}{\partial x^2} + \frac{1}{Pr}\cdot\frac{1}{y^k}\frac{\partial}{\partial y}\left(y^k\frac{\partial T}{\partial y}\right).
\end{aligned}\right\} \qquad (4.2)$$

When we allow the Reynolds number to tend to infinity and return to dimensional variables we finally obtain

$$\left.\begin{aligned}
&u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{1}{y^k}\frac{\partial}{\partial y}\left(y^k\frac{\partial u}{\partial y}\right),\\
&k\frac{w^2}{y} = \frac{1}{\rho}\frac{\partial p}{\partial y},\\
&u\frac{\partial \tilde{w}}{\partial x} + v\frac{\partial \tilde{w}}{\partial y} + \frac{v\tilde{w}}{y} = \nu\left\{\frac{1}{y^k}\frac{\partial}{\partial y}\left(y^k\frac{\partial \tilde{w}}{\partial y}\right) - \frac{\tilde{w}}{y^2}\right\},\\
&\frac{\partial}{\partial x}(y^k u) + \frac{\partial}{\partial y}(y^k v) = 0,\\
&u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = a\cdot\frac{1}{y^k}\frac{\partial}{\partial y}\left(y^k\frac{\partial T}{\partial y}\right).
\end{aligned}\right\} \qquad (4.3)$$

The system of L. Prandtl's Equations (4.3) obtained in this way is essentially simpler than System (4.1) of the Navier-Stokes equations. The simplification achieved is first of all due to the fact that the three equations of motion (for the velocity components u, v and w) were reduced to two (the first and the third) in the axisymmetric case and one (the first) in the case of a plane motion. The second equation is replaced by the fundamental law of the dynamics of a boundary layer in a plane-parallel motion, expressing the constancy of pressure perpendicular to the boundary layer. Moreover, the number of terms could be reduced in the right-hande sides of the first, third and fifth equations of System (4.3). These advantages also exist in more complex

cases of flow in a boundary layer, thus permitting a wide application of the boundary layer equations in hydodynamics and gasdynamics, particularly in the theory of jet flows.

In addition to this it should be stressed that the boundary layer equations hold strictly only in the case of very high (theoretically infinite) values of the Reynolds number, i.e., just for such conditions with which there is virtually no laminar motion. A solution of the problem of laminar jet motions has therefore mainly a general theoretical significance. Besides, as we shall show later by way of particular examples, the integrals of the boundary layer equations contain a certain singularity in a series of cases as already indicated by L. Prandtl [279].

Let us briefly consider the general methods of integrating the boundary layer equations applied when solving problems on the motion of jets of a viscous fluid.

First of all it should be noted that for laminar jets (as previously for exact solutions) the conception of the source jet maintains its applicability; in a first approximation it yields a self-similar solution without requiring detailed initial efflux conditions given beforehand. The problems (also comprising such which have no self-similar solutions) may then be investigated by means of two fundamental methods of the boundary layer theory, assuming finite or infinite thickness of the layer.

In the first case the boundary conditions for velocity, temperature, etc. are given for definite outer boundaries ($y = \delta$, $y = \delta_T$) of the dynamic and thermal boundary layers. Their thickness (δ and δ_T) and also a series of characteristic parameters (the thickness of displacement of momentum, heat content and the like) are introduced explicitly into the solution. The motion beyond the boundary layer is assumed to

be a potential motion.

In the second case the boundary layer is assumed to expane as if it were arbitrarily far away; the boundary conditions are given for $y = \infty$. The idea of this (asymptotically) infinite boundary layer is however, not in contradiction to the original idea of the boundary layer as a limited region in which the action of the forces of viscosity are chiefly localized. The solutions of the equation in fact given for the velocity, temperature, etc. are functions of the transverse coordinates which decrease so rapidly that at a finite and rather small distance which determines the effective thickness of the layer, the influence of the viscosity has virtually vanished. In this case we obtain a smooth asymptotic solution for the flow around the jet. The variables (u, T, etc.) and their derivatives with respect to the coordinates at the effective jet boundaries have no singularities.

Besides the direct integration of the differential equations of a (finite or asymptotic) boundary layer or their transformation to, e.g., a linear equation of the type of the heat-conduction equation (in coordinates according to Mises et al.), for the solution of problems on jet motions one also uses the method of integral relations and other approximation methods of calculating. They are all considered in detail in the monographs on the boundary layer theory [135, 155, 174, 212, etc.]. It is therefore expedient to illustrate the application of the various methods of solving jet problems immediately by way of examples; as to the details we refer to the sources mentioned above.

Note by the way that all these methods and also a series of others (semiempirical and empirical) are also used in the theory of turbulent jets.

4.2. THE AXISMMETRIC TWISTED SOURCE JET

Let us derive the problem of the expansion of an axisymmetric,

free and twisted jet of an incompressible viscous fluid. The jet is assumed to flow out of an unlimited space filled by a fluid of the same physical properties ("submerged" jet). In the general case we shall assume the temperature of the fluid in the jet different from the temperature of the medium. We shall further assume that the difference between the temperatures of jet and surrounding medium is so small that the density of the fluid as well as the viscosity and heat conduction coefficients which, in general, depend on the temperature (and the pressure) can be considered as constant. With these assumptions the temperature plays the part of some "coloring" of the fluid. Neglecting the heat of friction (and the radiation) the heat propagation equation is analogous to the diffusion equation for some (nonactive) impurity.

We start from the boundary layer equations for an sxisymmetric twisted source jet which therefore have the form [131]

$$\left.\begin{aligned}
&u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial y}\right),\\
&\frac{\rho w^2}{y}=\frac{\partial p}{\partial y},\\
&u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+\frac{vw}{y}=\nu\left\{\frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial w}{\partial y}\right)-\frac{w}{y^2}\right\},\\
&\frac{\partial}{\partial x}(yu)+\frac{\partial}{\partial y}(yv)=0,\\
&u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=a\frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial T}{\partial y}\right),
\end{aligned}\right\} \tag{4.4}$$

The boundary conditions for the asymptotic layer can be written in the form

$$\left.\begin{aligned}
&v=0,\ w=0,\ \frac{\partial u}{\partial y}=\frac{\partial T}{\partial y}=0 \quad \text{when} \quad y=0,\\
&u=w=\Delta T=p=0 \text{ when } y=\infty.
\end{aligned}\right\} \tag{4.5}$$

(The symbols ΔT and p denote the surplus temperature and pressure with respect to the constant values T_∞ and p_∞, the temperature and the pressure of the surrounding medium.)

As to the boundary conditions (4.5) and their analogues we shall remark in the following that solutions of problems obtained by means of the method of the asymptotic layer, in contrast to the boundary layer

of finite thickness, yield a smooth fit of the distributions of the (e.g. the velocity, the temperature, etc.) in the jet and in the surrounding medium. The solutions obtained within the framework of the theory of the asympotic boundary layer will therefore satisfy "automatically", for example, in the case considered, the equation

$$\frac{\partial u}{\partial y}=\frac{\partial w}{\partial y}=\frac{\partial T}{\partial y}=0 \text{ when } y=\infty.$$

These additional boundary conditions (the vanishing of the derivatives $\frac{\partial u}{\partial y}, \frac{\partial T}{\partial y}$ etc. at the outer boundary of the jet) can thus be used in deriving the integral relations or in other cases, when this is necessary for the estimation of total effects and the like, without overdefining the problem. Owing to this, the equality $\frac{\partial u}{\partial y}=0$ with $y=\infty$ or other analogous relations which are not used in the integration are sometimes written to the boundary conditions (which then seem to exceed the order of the differential equation).

In solutions obtained by the method of the finite boundary layer the number of boundary conditions is chosen in agreement with the necessity of determining, in the general case, the unknown values of the (dynamic, thermal) boundary layer coordinates from the solution. In this case, in particular in the thermal problem, the continuity at the boundary is very often violated even for the first derivative. E.g., in the thermal problem of the jet boundary $\frac{\partial T}{\partial y}\neq 0$ at the outer boundary and the heat flow displays a discontinuity.

For a source jet we shall, as usually, consider the self-similar solution which holds true at a sufficiently large distance from the orifice.

Just as in the exact solutions we introduce the integral conditions of conservation

$$J_x=2\pi\int_0^\infty y(p+\rho u^2)\,dy=\text{const}, \tag{4.6}$$

$$M_x = 2\pi \int_0^\infty \rho u w y^2 dy = \text{const}, \tag{4.7}$$

$$Q = 2\pi \int_0^\infty \rho c_p u \Delta T y dy = \text{const}. \tag{4.8}$$

The first and second of these conditions describe the conservation of momentum flux (in the projection on the Ox axis) and the angular momentum of the jet relative to the axis of symmetry, the third condition is the condition of conservation of the surplus heat content of the jet (in the absence of heat sources).

Recall the origin of the integral conditions. We multiply the first equation System (4.4) by y and the fourth by u and add them:

$$\frac{\partial}{\partial x}[y(p+\rho u^2)] + \frac{\partial}{\partial y}(\rho y u v) = \mu \frac{\partial}{\partial y}\left(y \frac{\partial u}{\partial y}\right).$$

Integrating this equation across the boundary layer from $y = 0$ to $y = \infty$ and taking the boundary conditions (4.5) into account, we obtain the integral invariant (4.6).

Let us now multiply the equation of continuity by w and the third equation of the system (r.r) by y. We then obtain as the result of their multiplication

$$\frac{\partial}{\partial x}(yuw) + \frac{\partial}{\partial y}(yvw) + vw = \nu \frac{\partial}{\partial y}\left[\frac{1}{y}\frac{\partial}{\partial y}(yw)\right].$$

The last equation will again be multiplied by y and rewritten in the form

$$\frac{\partial}{\partial x}(y^2 uw) + \frac{\partial}{\partial y}(y^2 vw) = \nu \frac{\partial}{\partial y}\left[y \frac{\partial}{\partial y}(yw) - 2yw\right].$$

An integration of this equation with respect to y within the limits from 0 to ∞, taking the oundary conditions (4.5) into account, yields the integral conservation condition (4.7).

In an analogous way we arrive at the invariant (4.8): the energy equation is multiplied by y and added to the continuity equation which has been multiplied preliminarily by ΔT. A subsequant integration

yields (4.8). In a series of cases the nonvanishing value of the integral invariant can be given such (e.g., $M_x \neq 0$ with $w(0) = w(\infty) = 0$) that a nontrivial (nonvanishing) solution can be obtained.

A. Solution by the method of the asymptotic layer. In order to obtain a self-similar solution we transform the system Equations (4.4) to a system of ordinary differential equations by means of a substitution of variables; we shall agree in calling this a self-similar transformation. Let us introduce the following power functions*:

$$u_m = Ax^{\alpha},\ w_m = Cx^{\varepsilon},\ p_m = \rho Dx^{\delta},\ \Delta T_m = \Gamma x^{\gamma},\ \varphi = Byx^{\beta}, \qquad (4.9)$$

and the denotations

$$\frac{u}{u_m} = \frac{F'(\varphi)}{\varphi},\quad \frac{w}{w_m} = \Phi(\varphi),\quad \frac{p}{p_m} = P(\varphi),\quad \frac{\Delta T}{\Delta T_m} = \theta(\varphi)$$

($u_m, \Delta T_m, p_m$ are the values of the velocity, the surplus temperatures and the pressure at the axis of the jet, w_m is the maximum value of the peripheral velocity; the prime here and in the following indicates differentiation with respect to the variable φ).

Taking into account that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial \varphi}{\partial x}\frac{d}{d\varphi} = \frac{\partial}{\partial \xi} + \beta\frac{\varphi}{x}\frac{d}{d\varphi},\quad \frac{\partial}{\partial y} = \frac{\partial \varphi}{\partial y}\frac{d}{d\varphi} = Bx^{\beta}\frac{d}{d\varphi},$$

we transform all terms if Eqs (4.4) to the new independent variable $\xi \equiv x,\ \varphi$:

$$\frac{\partial u}{\partial x} = Ax^{\alpha-1}\left[\alpha\frac{F'}{\varphi} + \beta\varphi\left(\frac{F'}{\varphi}\right)'\right],$$

$$\frac{\partial u}{\partial y} = ABx^{\alpha+\beta}\left(\frac{F'}{\varphi}\right)',$$

$$\frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial y}\right) = AB^2x^{\alpha+2\beta}\frac{1}{\varphi}\left[\varphi\left(\frac{F'}{\varphi}\right)'\right]',$$

$$v = -\frac{1}{y}\int_0^y \frac{\partial}{\partial x}(yu)\,dy = -\frac{A}{B}x^{\alpha-\beta-1}\left[(\alpha-2\beta)\frac{F}{\varphi} + \beta F'\right],$$

$$\frac{\partial w}{\partial x} = Cx^{\varepsilon-1}(\varepsilon\Phi + \beta\varphi\Phi'),$$

$$\frac{\partial w}{\partial y} = BCx^{\beta+\varepsilon}\Phi',$$

$$\frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial w}{\partial y}\right) = B^2 C x^{2\beta+\varepsilon}\frac{1}{\varphi}(\varphi\Phi')',$$

$$\frac{\partial T}{\partial x} = \Gamma x^{\gamma-1}(\gamma\theta + \beta\varphi\theta'),$$

$$\frac{\partial T}{\partial y} = B\Gamma x^{\beta+\gamma}\theta',$$

$$\frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial T}{\partial y}\right) = B^2\Gamma x^{2\beta+\gamma}\frac{1}{\varphi}(\varphi\theta')',$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y} = BDx^{\beta+\delta}P'.$$

Substituting the reduced values in the initial system of equations of this system the term $-\frac{1}{\rho}\frac{\partial p}{\partial x}$ (which, as shown below, decreases with the growth of x much more rapidly than the other terms of the equation*), we obtain after a transformation

$$\left.\begin{aligned}
&\frac{1}{\varphi}\left[\varphi\left(\frac{F'}{\varphi}\right)'\right]' + \frac{A}{\nu B^2}x^{\alpha-2\beta-1}\left[(\alpha-2\beta)\frac{F}{\varphi}\left(\frac{F'}{\varphi}\right)' - \alpha\left(\frac{F'}{\varphi}\right)^2\right] = 0\\
&P' = \frac{C^2}{D}x^{2\varepsilon-\delta}\,\frac{\Phi^2}{\varphi},\\
&\frac{(\varphi\Phi')'}{\varphi} - \frac{\Phi}{\varphi^2} + \frac{A}{\nu B^2}x^{\alpha-2\beta-1}\Big[(\alpha-2\beta)\left(\Phi' + \frac{\Phi}{\varphi}\right)\frac{F}{\varphi} + {}\\
&\qquad\qquad\qquad\qquad + (\beta-\varepsilon)\frac{F'\Phi}{\varphi}\Big] = 0,\\
&\frac{(\varphi\theta')'}{\varphi} + \Pr\frac{A}{\nu B^2}x^{\alpha-2\beta-1}\left[(\alpha-2\beta)\frac{F\theta'}{\varphi} - \gamma\frac{F'\theta}{\varphi}\right] = 0.
\end{aligned}\right\}\quad (4.10)$$

We require that the equations obtained are independent of the variable x. In this case we have

$$\beta = \frac{\alpha-1}{2},\quad \delta = 2\varepsilon, \qquad (4.11)$$

and in the equations three "self-similarity constants" will remain: α, ε and γ:

$$\frac{1}{\varphi}\left[\varphi\left(\frac{F'}{\varphi}\right)'\right]' + \frac{A}{\nu B^2}\left[\frac{F}{\varphi}\left(\frac{F'}{\varphi}\right)' - \alpha\left(\frac{F'}{\varphi}\right)^2\right] = 0, \qquad (4.12)$$

$$P' = \frac{C^2}{D}\frac{\Phi^2}{\varphi}, \qquad (4.13)$$

$$\frac{(\varphi\Phi')'}{\varphi} - \frac{\Phi}{\varphi^2} + \frac{A}{\nu B^2}\left[\frac{F}{\varphi}\left(\Phi' + \frac{\Phi}{\varphi}\right) + \frac{1}{2}(\alpha - 2\varepsilon - 1)\frac{F'\Phi}{\varphi}\right] = 0, \qquad (4.14)$$

$$\frac{(\varphi\theta')'}{\varphi} + \Pr\frac{A}{\nu B^2}\left(\frac{F\theta'}{\varphi} - \gamma\frac{F'\theta}{\varphi}\right) = 0. \qquad (4.15)$$

Let us now use the integral Conditions (4.6)-(4.8) (by virtue of what has been shown above, in the first of them the pressure will not enter the integrand) in order to determine the constants α, ε and γ.

We rewrite (4.6)-(4.8) in the form

$$2\pi\rho \frac{A^2}{B^2} x^{2(\alpha-\beta)} \int_0^\infty \left(\frac{F'}{\varphi}\right)^2 \varphi\, d\varphi = J_x, \tag{4.16}$$

$$2\pi\rho \frac{AC}{B^3} x^{\alpha-3\beta+\varepsilon} \int_0^\infty F'\Phi\varphi\, d\varphi = M_x, \tag{4.17}$$

$$2\pi\rho c_p \frac{A\Gamma}{B^2} x^{\alpha-2\beta+\gamma} \int_0^\infty F'\theta\, d\varphi = Q, \tag{4.18}$$

and obtain

$$\alpha = \beta,\ \varepsilon = 3\beta - \alpha = 2\alpha,\ \gamma = 2\beta - \alpha = -1.$$

Taking Eq. (4.11) into account we finally obtain

$$\alpha = \beta = \gamma = -1,\ \varepsilon = -2,\ \delta = -4. \tag{4.19}$$

We find from the same integral conditions

$$\frac{A^2}{B^2} = \frac{J_x}{2\pi\rho J_1},\quad \frac{AC}{B^3} = \frac{M_x}{2\pi\rho J_2},\quad \frac{A\Gamma}{B^2} = \frac{Q}{2\pi\rho c_p J_3}, \tag{4.20}$$

where the symbols J_1, J_2 and J_3 denote the definite integrals

$$J_1 = \int_0^\infty \left(\frac{F'}{\varphi}\right)^2 \varphi\, d\varphi = \frac{4}{3},\quad J_2 = \int_0^\infty F'\Phi\varphi\, d\varphi = \frac{16}{3},$$
$$J_3 = \int_0^\infty F'\theta\, d\varphi = \frac{4}{2Pr+1}, \tag{4.20a}$$

whose values, which are given here, have been obtained from the solution of the problem derived in the following.

Returning to Eqs. (4.12)-(4.15) we choose arbitrary values for the ratios of the constants:

$$\frac{A}{\nu B^2} = 1,\ \frac{C^2}{D} = \frac{3}{4}. \tag{4.21}$$

Taking Eq. (4.20a) into account, we obtain from (4.20) and (4.21)

$$A = \frac{3J_x}{8\pi\rho\nu},\quad B = \sqrt{\frac{3J_x}{8\pi\rho\nu^2}},\quad C = \frac{3M_x}{32\pi\rho\nu^2}\sqrt{\frac{3J_x}{8\pi\rho}},$$
$$D = \frac{9}{2048}\frac{M_x^2 J_x}{\pi^3\rho^3\nu^4};\quad \Gamma = \frac{(2Pr+1)Q}{8\pi\rho\nu c_p}.$$

Finally we shall have the equations

$$\left(F'' - \frac{F'}{\varphi}\right)' + \left(\frac{FF'}{\varphi}\right)' = 0, \tag{4.22}$$

$$P' = \frac{\Phi^2}{\varphi}, \tag{4.23}$$

$$\Phi'' + \frac{1+F}{\varphi}\Phi' + \frac{\varphi F' + F - 1}{\varphi^2}\Phi = 0, \tag{4.24}$$

$$(\varphi\theta')' + Pr(F\theta)' = 0 \tag{4.25}$$

with the boundary conditions

$$\left.\begin{aligned} &\frac{F'}{\varphi} = 1,\ \frac{F}{\varphi} = 0,\ \Phi = \theta' = 0 \text{ when } \varphi = 0,\\ &\frac{F'}{\varphi} = 0,\ \Phi = \theta = P = 0 \text{ when } \varphi = \infty. \end{aligned}\right\} \tag{4.26}$$

As always in the case of an incompressible fluid with constant physical properties, the dynamic problem is solved separately from the thermal problem.

Integrating Eq. (4.22) we obtain

$$F'' - \frac{F'}{\varphi} + \frac{FF'}{\varphi} = C_1.$$

From the conditions of symmetry we obtain $C_1 = 0$ on the axis of the jet. Repeated integration yields

$$\varphi F' + \frac{1}{2}(F - 4)F = C_2,$$

where $C_2 = 0$. A third integration, taking the boundary conditions into account, leads us to the equality

$$F = \frac{4C_3\varphi^2}{1 + C_3\varphi^2},$$

where $C_3 = 1/8$. Hence we fine

$$F = \frac{\frac{1}{2}\varphi^2}{1 + \frac{1}{8}\varphi^2},\quad \frac{F'}{\varphi} = \frac{1}{\left(1 + \frac{1}{8}\varphi^2\right)^2}. \tag{4.27}$$

Let us now determine the law of variation (increase) of the fluid's flow rate in the jet

$$G = 2\pi\int_0^\infty \rho u y\, dy = 2\pi\rho\frac{A}{B^2}x[F(\infty) - F(0)],$$

or

$$G = 8\pi\mu x,$$

and the law of variation (decrease) of the kinetic energy flux

$$E = \pi\int_0^\infty \rho u^2 y\, dy = \pi\rho\frac{A^2}{B^2}\frac{1}{x}\int_0^\infty \frac{F'^2}{\varphi^2}d\varphi,$$

or

$$E = \frac{9J_x^2}{64\pi\mu}\frac{1}{x}\int\limits_0^\infty \frac{F'^2}{\varphi^3}\,d\varphi = \frac{\mathrm{const}}{x}.$$

Turning to the integration of Eq. (4.24) we must remark first of all that it may be rewritten in the form

$$[\varphi^2\Phi' - \varphi(1-F)\Phi]' = 0,$$

and hence we obtain

$$\varphi^2\Phi' - \varphi(1-F)\Phi = C_4.$$

From the boundary conditions (4.26) follows the equality $C_4 = 0$, so that in order to obtain the function Φ we obtain the equation

$$\frac{d\Phi}{\Phi} = \frac{1-F}{\varphi}\,d\varphi,$$

from which, taking Eq. (4.27) into account, we finally obtain

$$\Phi(\varphi) = \frac{\varphi}{\left(1+\frac{1}{8}\varphi^2\right)^4} \tag{4.28}$$

(the constant of integration which is a factor in the right-hand side of the last equation is assumed to be equal to unity; this is possible since the expression for the peripheral velocity contains an arbitrary constant C).

Using the solution of the dynamic problem, we integrate the temperature Equation (4.25), taking the boundary condition (4.26) into account. We have

$$\varphi\theta' + \mathrm{Pr}\,F\theta = 0.$$

Rewriting the latter equation in the form

$$\frac{d\theta}{\theta} = -\mathrm{Pr}\,\frac{F}{\varphi}\,d\varphi$$

or, determining the function F from the equation (see above)

$$F'' - \frac{F'}{\varphi} + \frac{FF'}{\varphi} = 0,$$

we obtain

$$\frac{d\theta}{\theta} = \mathrm{Pr}\,\frac{d\left(\frac{F'}{\varphi}\right)}{\left(\frac{F'}{\varphi}\right)}.$$

Hence (taking into account that $\theta(0) = 1$) we find

$$\theta = \left(\frac{F'}{\varphi}\right)^{Pr} = \frac{1}{\left(1+\frac{1}{8}\varphi^2\right)^{2Pr}}. \tag{4.29}$$

Note that the final result of the solution of our thermal problem can be represented in the form

$$\frac{\Delta T}{\Delta T_m} = \left(\frac{u}{u_m}\right)^{Pr}. \tag{4.30}$$

This equation may be interpreted clearly from the point of view of the thickness ratio of the dynamic and the thermal boundary layers. With a Prandtl number $Pr = 1$ ($\nu = a$) the effective thicknesses of the layers agree with one another; with $\nu<a$ (or $\nu>a$) the dynamic layer will be narrower (or broader) than the thermal. As regards the values of the effective layer thicknesses they are easy to determine if the desired degree of accuracy is given.

Let us consider for comparison two values of the relative velocity and temperature: the value $\frac{u}{u_m} = \frac{\Delta T}{\Delta T_m} = 0{,}5,$ characterizing the middle part of the distribution, and a definite external boundary, e.g., $\frac{u}{u_m} = \frac{\Delta T}{\Delta T_m} = 10^{-2}$. The numerical values of the nondimensional variables corresponding to the chosen values of velocity and temperature for three values of the Prandtl number are compiled in the talbe

TABLE A

$\frac{u}{u_m}$, $\frac{\Delta T}{\Delta T_m}$, $\left(\frac{F'}{\varphi}\right)_\delta$, θ_{δ_T}	φ_δ	φ_{δ_T}		
		$Pr=0{,}5$	$Pr=1{,}0$	$Pr=2{,}0$
$\frac{1}{2}$	1,82	2,83	1,82	1,27
10^{-2}	8,49	28,14	8,49	4,16

The above choice of the boundary values of $\left(\frac{F'}{\varphi}\right)_\delta$ and θ_{δ_T} is of course arbitrary, it can illustrate, however, the rate of decline and the relative courses of velocity and temperature in the cross sections of an asymptotic ("infinite") boundary layer.

The results of the obtained solution of the problem are illustrated in Figs. 4.1 and 4.2.

The mathematical expressions for the problem considered are contained in Table 7.1 given below, together with the results of solutions of other problems dealing with the expansion of laminar jets of an incompressible fluid.

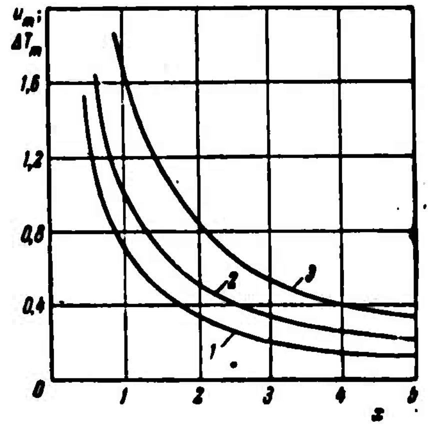

Fig. 4.1. Variation of velocity and surplus temperature along the axis of a round jet (in arbitrary coordinates). 1. ΔT_m; Pr = 0.5; 2. u_m; ΔT_m; Pr = 1.0; 3. ΔT_m; Pr = 2.0.

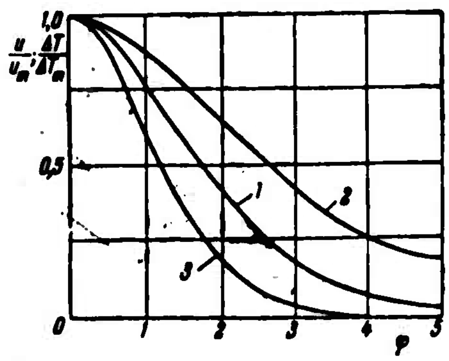

Fig. 4.2. Universal distributions of velocity and surplus temperature in cross sections of a round jet. 1. u/u_m; $\Delta T/\Delta T_m$; Pr = 1.0; 2. $\Delta T/\Delta T_m$; Pr = 0.5; 3. $\Delta T/\Delta T_m$; Pr = 2.0.

<u>B. Approximate solution.</u> In spite of the fact that the problem of the axisymmetric source jet can be solved by quadratures, we give for comparison an approximate solution of the same problem, which is based on the integral method of the boundary layer of finite thickness.

For the sake of simplicity we consider the case of a nontwisted

jet and, without integrating the equations

$$\left.\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}&=\frac{\nu}{y}\frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial y}\right),\\ \frac{\partial}{\partial x}(yu)+\frac{\partial}{\partial y}(yv)&=0,\end{aligned}\right\} \tag{4.31}$$

we choose the velocity distribution on the basis of the "smoothness" to be expected for the solution in the form of a polynomial of e.g., third degree

$$u=a_0+a_1y+a_2y^2+a_3y^3. \tag{4.32}$$

Satisfying the boundary conditions on the axis of the jet

$$\frac{\partial u}{\partial y}=0,\ \frac{\nu}{y}\frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial y}\right)=u_m\frac{du_m}{dx}\ \text{when}\ y=0$$

(the second condition is obtained with $v = 0$ from the first equation of System (4.31) and at the outer boundary of the jet

$$u=0,\ \frac{\partial u}{\partial y}=0\ \text{when}\ y=\delta,$$

we obtain four linear equation in order to determine the coefficients a_0-a_3. The values found for a_0-a_3 are compiled in the following table:

a_0	a_1	a_2	a_3
$-\frac{\delta^2}{12\nu}u_m\frac{du_m}{dx}$	0	$\frac{u_m}{4\nu}\frac{du_m}{dx}$	$-\frac{u_m}{6\nu\delta}\frac{du_m}{dx}$

The velocity distribution for these values of the coefficients is determined by the expression

$$u=\frac{\delta^2}{12\nu}u_m\frac{du_m}{dx}\left[-1+3\left(\frac{y}{\delta}\right)^2-2\left(\frac{y}{\delta}\right)^3\right]. \tag{4.33}$$

With $y = 0$ we find from Eq. (4.33) an equation which links the velocity on the jet axis, $u_m = u_m(x)$ and the boundary layer thickness $\delta = \delta(x)$:

$$\frac{du_m}{dx}=-\frac{12\nu}{\delta^2}. \tag{4.34}$$

Taking this relation into account, we can represent the velocity

distribution in a nondimensional form $\left(\bar{u}=\frac{u}{u_m},\ \bar{y}=\frac{y}{\delta}\right)$:

$$\bar{u}=1-3\bar{y}^2+2\bar{y}^3. \qquad (4.35)$$

Using the integral condition of momentum conservation

$$J_x=2\pi\int_0^\delta \rho u^2 y\, dy,$$

which we rewrite in the form

$$J_x=2\pi\rho\delta^2 u_m^2\int_0^1 \bar{u}^2\bar{y}\, d\bar{y},$$

we obtain another equation linking u_m and δ:

$$\delta^2 u_m^2=\frac{J_x}{2\pi\rho J}\quad\left(J=\int_0^1 \bar{u}^2\bar{y}\, d\bar{y}=\frac{3}{35}\right). \qquad (4.36)$$

From Eqs. (4.34) and (4.36) we obtain

$$\frac{du_m}{dx}=-\frac{72}{35}\frac{\pi\rho\nu}{J_x}u_m^2,$$

which, when we take the condition $\delta = 0$, $u_m = \infty$ if $x = 0$ into account, yields the following expression for the velocity at the jet axis:

$$u_m=\frac{35}{72}\frac{J_x}{\pi\rho\nu}\frac{1}{x} \qquad (4.37)$$

and for the boundary layer thickness

$$\delta=12\nu x\sqrt{\frac{6\pi\rho}{35J_x}}. \qquad (4.38)$$

Let us now analyze the solution obtained. First of all it should be noted that it is also self-similar with the same values of the constants of self-similarity $\alpha = \beta = -1$ as above. As regards the constants A and B entering the transformation formulas $u_m = Ax^{\alpha}$, $\varphi = Byx^{\beta}$, the first of them, A, which characterizes the longitudinal damping of the jet and is obtained from the approximate solution as equal to $A' = \frac{35}{72}\frac{J_x}{\pi\rho\nu}\approx 0{,}486\frac{J_x}{\pi\rho\nu}$, exceeds the exact value $A=0{,}375\frac{J_x}{\pi\rho\nu}$ by about 30%.

In contrast to this, the value of the constant $B'\approx 0{,}201\sqrt{\frac{J_x}{\pi\rho\nu^2}}$ in the approximate solution* is more than three times smaller than the exact value $B=0{,}612\sqrt{\frac{J_x}{\pi\rho\nu^2}}$.

The relative similarity of the values of A' and A and the great difference between B' and B for the approximate and the exact solutions is easy to explain. The constant A' is determined by means of the integral condition of momentum conservation in which the velocity distribution only exerts an indirect influence (through the value of the integral J). The constant B' depends immediately on the shape of the distribution. The latter, determined by the approximate solution (4.35), corresponds to the linear differential equation of the form $\left(\frac{F'}{\varphi}\right)' = C + D\varphi$ instead of the nonlinear differential equation of the form

$$\left[F'' - \frac{F'(1-F)}{\varphi}\right]' = 0.$$

In other words, the approximate solution (4.35) obtained is no integral of the initial system of differential equations.

The results of the calculations obtained from the exact and the approximate formulas differ essentially in the outer parts of the jet, the approximate solution results in a more rapid velocity drop in the jet cross section.

This way of calculation may of course also be used in order to solve the thermal problem. Let us show this by way of the same example of an axisymmetric jet. We shall here consider the dynamic problem solved (by the approximate method) and, in order to vary the case, not the temperature distribution in the cross section but the heat flux distribution $q = -\lambda \frac{\partial \Delta T}{\partial y}$. given in the form of a polynomial.

We shall use, just as in the case of the dynamic problem of velocity, a four-term expression for the temperature

$$-\frac{q}{\lambda} = \frac{\partial \Delta T}{\partial y} = b_0 + b_1 y + b_2 y^2. \tag{4.39}$$

The boundary conditions for the heat flux:

$$q = 0 \text{ if } y = 0 \text{ and } y = \delta_T. \tag{4.40}$$

Moreover, we find from (4.4) for the condition on the jet axis

$$u_m \frac{d\Delta T_m}{dx} = \frac{a}{y} \frac{\partial}{\partial y}\left(y \frac{\partial \Delta T}{\partial y}\right)\Big|_{y=0}. \qquad (4.41)$$

Finally we also take the equation

$$\Delta T = 0 \text{ if } y = \delta_T \qquad (4.42)$$

into account which holds true at the boundary of the thermal layer.

Without giving the simple transformations which are analogous to the preceding ones we write the final expression for the temperature distribution:

$$\theta = \frac{\Delta T}{\Delta T_m} = 1 - 3\tilde{y}^2 + 2\tilde{y}^3 \left(\tilde{y} = \frac{y}{\delta_T}\right). \qquad (4.43)$$

It is evident that the relative distributions u/u_m and $\Delta T/\Delta T_m$ in the jet cross section coincide if in each of them its scale unit (δ or δ_T) for the coordinate y is introduced. With $\delta = \delta_T$ they also coincide completely in the physical plane x, y.

As shown in the exact solution, the equality of the effective thicknesses of the dynamic and thermal layers $\delta = \delta_T$ corresponds to $Pr = 1$. Accordingly, with $Pr \approx 1$ we may obtain $\delta_T \approx \delta$. To calculate $\Delta T_m = \Delta T_m(x)$ we use under this supposition $\delta = \delta_T$ the integral condition of conservation of the excessive heat content

$$Q = 2\pi \int_0^\delta \rho c_p u \Delta T y \, dy = \text{const.} \qquad (4.44)$$

Substituting here the values of u from (4.35) and ΔT from (4.43) we obtain after a recalculation

$$\Delta T_m = \Gamma' \frac{1}{x}, \quad \Gamma' = \frac{35Q}{72\pi\lambda} = 0{,}486 \frac{Q}{\pi\lambda}.$$

The value obtained for the constant Γ' is relatively similar (with $Pr = 1$) to the exact value $\Gamma = 0{,}375 \frac{Q}{\pi\lambda}$. Here, as a consequence of the assumption $\delta = \delta_T$, $\frac{\Gamma'}{\Gamma} = \frac{A'}{A}$.

In the theory of layers of finite thickness the assumption of equal thickness of dynamic and thermal boundary layers is widespread. In the case of Prandtl numbers different from unity it is connected with an essential distortion of both the quantitative and the qualitative image of the process. It can be tried to improve the approximate results a little for the range of Pr > 1 by an independent determination of the thickness δ_T of the thermal boundary layer. In this zone,* obviously, $\delta_T < \delta$.

The independent quantity δ_T introduced above may be determined from Eq. (4.44) in which the upper limit of the integral is replaced by δ_T and from the condition at the jet axis in the form

$$\Delta T_m = -\frac{u_m \delta_T^2}{12a}\frac{d\Delta T_m}{dx},$$

which is analogous to Condition (4.34) for the velocity on the jet axis.

From the equation

$$Q = 2\pi \int_0^{\delta_T} \rho c_p u \,\Delta T y\, dy$$

we obtain

$$Q = 0{,}3\pi\rho c_p u_m \Delta T_m \delta_T^2 \left(1 - \frac{5}{7}\frac{\delta_T^2}{\delta^2} + \frac{2}{7}\frac{\delta_T^3}{\delta^3}\right).$$

Let us restrict ourselves to the case of $\delta_T \ll \delta$, which corresponds to a droppable liquid (Pr >> 1) and drop the second and third terms in the parentheses of the above expression. We than have

$$\Delta T_m \delta_T^2 = \frac{Q}{0{,}3\pi\rho c_p u_m}.$$

Using the above equations which link the quantities ΔT_m and δ_T and also u_m and δ, we finally arrive at

$$\Delta T_m = \frac{5}{18}\frac{Q}{\pi\lambda}\frac{1}{x} = \frac{4}{7}\Gamma'\frac{1}{x}, \quad \delta_T = 4{,}968\sqrt{\frac{\pi\rho a}{J_x}}\,x.$$

The latter formula may be written in a simpler form which is char-

acteristic of solutions obtained by the method of finite thickness:

$$\frac{\delta_T}{\delta} = \frac{1}{\sqrt{Pr}}.$$

It should be mentioned finally that in the theory of laminar jets the method of the layer of finite thickness has not only no advantages compared to the method of the asymptotic layer but is also more accurate than it as we showed above. With the modern development of the technique of computation it can in general not be considered justified to apply the method of the layer of finite thickness in the theory of the free laminar layer (this will of course also hold true for the theory of the laminar boundary layer as a whole).

Otherwise it is necessary to estimate the solutions of problems on the expansion of turbulent jets with the help of the method of the layer of finite thickness or general integral relations [75, 77, 304, etc.]. In the first place, the empirical constants introduced when solving the problem of the expansion of turbulent jets may comprise the lack of mathematical "binding." Secondly, these methods permit an arbitrary extension of the number of problems yielding to approximate calculation, also including several nonself-similar flows.

4.3. THE PLANE SOURCE JET

The methods discussed above which are used to determine self-similar solutions may also be used in investigations of the motion produced by a plane source-jet. Taking into account that such a solution is analogous to the preceding, we can summarize its representation, developing some problems specific of plane jets in greater detail.

The initial system of equations of the dynamic problem

$$\left.\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \nu\frac{\partial^2 u}{\partial y^2}, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned}\right\} \qquad (4.45)$$

with the boundary conditions

$$\frac{\partial u}{\partial y}=0, \quad v=0 \text{ if } y=0, \quad u=0 \quad \text{if } y=\pm\infty, \tag{4.46}$$

can be reduced by means of the self-similarity transformations

$$\frac{u}{u_m}=F'(\varphi), \quad u_m=Ax^\alpha, \quad \varphi=Byx^\beta$$

to an ordinary differential equation (with $A/B^2 = 6\nu$ and the values of the constants α and β of self-similarity determined below)

$$F''' + 2(FF'' + F'^2) = 0 \tag{4.47}$$

with the boundary conditions

$$F=0, \quad F'=1 \text{ if } \varphi=0, \quad F'=0, \quad \text{if } \varphi=\pm\infty. \tag{4.48}$$

The solution of this equation reads

$$F=\operatorname{th}\varphi, \quad F'=\frac{1}{\operatorname{ch}^2\varphi}=1-\operatorname{th}^2\varphi. \tag{4.49}$$

The values of the constants A, B and the constants α and β of self-similarity (which were determined by means of the integral condition of conservation of momentum of the jet $J_x=\int_{-\infty}^{+\infty}\rho u^2\,dy=\text{const}$) are equal to

$$A=\frac{1}{2}\sqrt[3]{\frac{3J_x^2}{4\rho^2\nu}}, \quad B=\frac{1}{2}\sqrt[3]{\frac{J_x}{6\rho\nu^2}}, \quad \alpha=-\frac{1}{3}, \quad \beta=-\frac{2}{3}.$$

Figure 4.3 shows the universal velocity distribution in the cross section of a plane jet in the coordinates $\frac{u}{u_m}=f(\varphi/\varphi_{1/2})$, where $\frac{\varphi}{\varphi_{1/2}}\equiv\frac{y}{y_{1/2}}$; as in other cases the abscissa $y = y_{\frac{1}{2}}$ corresponds to an ordinate value of $u = u_m/2$.

In this way the method of the asymptotic layer permits a simple solution of the problem. We shall therefore not give the approximate solution which can be obtained by the method of the finite layer, analogously as in the case of the axisymmetric jet.

Let us now turn to the thermal problem. We shall derive a solution

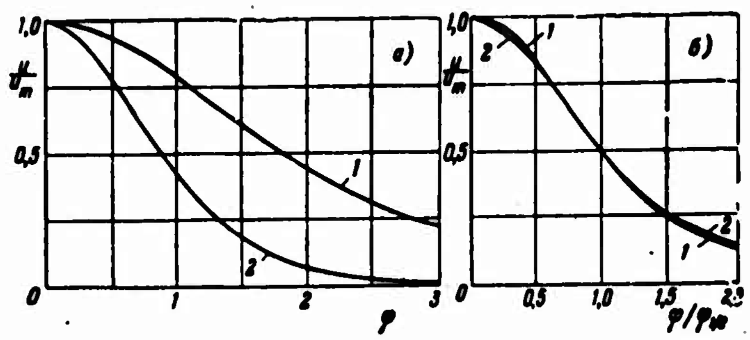

Fig. 4.3. Universal velocity distributions in jet cross sections. 1) Round jet; 2) plane jet.

for two types of boundary conditions for the temperature: such which are similar to the boundary conditions for the velocity (the values of the temperature are the same on either side of the jet) and such without this similarity (the temperature values at the outer boundaries of the jet are different). An analogous problem was considered in the first part for a fan-type jet.

Let us add to the system of equations (4.45) whose solution is assumed known the equation of heat transfer

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=a\frac{\partial^2 T}{\partial y^2} \tag{4.50}$$

and the two variants of boundary conditions:

a)
$$\left.\begin{aligned}\frac{\partial T}{\partial y}&=0 \text{ if } y=0,\\ T&=T_\infty \text{ if } y=\pm\infty\end{aligned}\right\} \tag{4.51a}$$

and

b)
$$\left.\begin{aligned}T&=T_1 \text{ if } y=+\infty,\\ T&=T_2 \text{ if } y=-\infty.\end{aligned}\right\} \tag{4.51b}$$

We introduce the self-similarity transformation:

a) for a symmetrical thermal boundary layer

$$T-T_\infty=(T_m-T_\infty)\theta_s(\varphi),\quad T_m-T_\infty=\Gamma x^\gamma; \tag{4.52a}$$

b) for an "asymmetric" thermal boundary layer, on the basis of physical considerations we assume [56] that at a great distance from

the source

$$\frac{T-T_2}{T_1-T_2}=\theta_s(\varphi) \quad \text{(i.e., } \gamma = 0\text{)}. \tag{4.52b}$$

In the first case, using the integral condition of conservation of the surplus heat content

$$Q=\int_{-\infty}^{+\infty}\rho c_p u\,(T-T_\infty)\,dy=\text{const}, \tag{4.53}$$

we obtain the value $\gamma = -1/3$, and Eq. (4.50) can then be written in the form

$$\theta_a''+2\Pr(F\theta_a)'=0 \tag{4.54}$$

with the boundary conditions

$$\begin{aligned}\theta_a'&=0 \quad \text{if} \quad \varphi=0,\\ \theta_a&=0 \quad \text{if} \quad \varphi=\pm\infty.\end{aligned} \tag{4.55}$$

The solution of Eq. (4.54), with the boundary conditions (4.55) and the integral condition (4.53) taken into account, can be written in the form

$$\theta_a=\frac{1}{(\operatorname{ch}\varphi)^{2\Pr}}, \qquad \Gamma=\frac{B}{A}\frac{Q}{\rho c_p}\left[\int_{-\infty}^{+\infty}(F')^{\Pr+1}\,d\varphi\right]^{-1}. \tag{4.56}$$

As also in the case of the axisymmetric jet, the solution of the thermal problem in the case of similar boundary conditions for velocity and temperature can be represented in the form

$$\frac{\Delta T}{\Delta T_m}=\frac{T-T_\infty}{T_m-T_\infty}=\left(\frac{u}{u_m}\right)^{\Pr}. \tag{4.57}$$

With Pr = 1 the universal velocity and temperature distributions coincide; with Pr ≠ 1 we have the correspondence between the effective thicknesses of the dynamic and the thermal boundary layers mentioned above.

In the second case, Eq. (4.50) and the boundary conditions (4.51b) can be rewritten in the form

$$\theta_3'' + 2\mathrm{Pr}F\theta_3' = 0, \tag{4.58}$$

$$\left.\begin{array}{l}\theta_3 = 1 \text{ if } \varphi = +\infty,\\ \theta_3 = 0 \text{ if } \varphi = -\infty.\end{array}\right\} \tag{4.59}$$

The solution of Eq. (4.58) with these boundary conditions and the equation $F'' + 2FF' = 0$ from the dynamic problem will read

$$\theta_3 = \frac{T - T_2}{T_1 - T_2} = C_0 \int\limits_{-\infty}^{\varphi} (\operatorname{ch}\varphi)^{-2\mathrm{Pr}}\, d\varphi, \quad C_0 = \left[\int\limits_{-\infty}^{+\infty} (\operatorname{ch}\varphi)^{-2\mathrm{Pr}}\, d\varphi\right]^{-1}. \tag{4.60}$$

In the simplest particular case of Pr = 1 the constant of integration is C_0 = 0.5 and

$$\theta_{\mathrm{Pr}=1} = \frac{1 + \operatorname{th}\varphi}{2}.$$

Let us give the numerical values of the constant C_0 for several values of the Prandtl number:

Pr	0,5	0,75	1	2
C_0 (Pr)	0,36	0,44	0,50	0,75

The corresponding distributions of the nondimensional temperature are given in Fig. 4.4.

Our example of asymmetric boundary conditions for the temperature and other analogous examples are interesting as they provide a scheme of the jet mixing in the interface of two volumes filled with media of different properties, e.g., two different gases or liquids. The development of a jet boundary layer along the plane interface of these volumes intensifies the heat and mass exchange (which is particularly essential in the case of turbulent mixing).

For the intensity characteristic of heat transfer through the interface y = 0 we introduce, as this is usually done, a nondimensional parameter, the Nusselt number $\mathrm{Nu}_x = \alpha x/\lambda$, which is a function of the Reynolds number $\mathrm{Re}_x = (J_x \cdot x)/\rho\nu^2$ characteristic of the problem, α being the heat transfer coefficient and λ the heat-conduction coefficient,

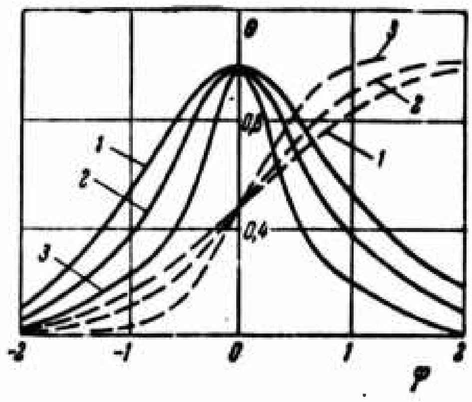

Fig. 4.4. Temperature distribution in cross sections of a plane jet. —) $T_1 = T_2 = T_\infty$; – – –) $T_1 \neq T_2$. 1) Pr = 0.5; 2) Pr = 1.0; 3) Pr = 2.0.

$$J_x = \int_{-\infty}^{+\infty} \rho u^2 \, dy = \text{const}$$

is the total momentum of the jet. The local Nusselt number is

$$\mathrm{Nu}_x = \frac{qx}{\lambda (T_1 - T_2)}, \quad \text{where} \quad q = -\lambda \frac{\partial T}{\partial y}\Big|_{y=0},$$

or, after a transformation,

$$\mathrm{Nu}_x = \frac{\partial \theta_\delta}{\partial y}\Big|_{y=0} \cdot x = Bx^{\beta+1} \theta'_\delta \big|_{\varphi=0} = BC_0 x^{1/3},$$

where the values of the constants B and C_0 are given above.

Since $B = \frac{1}{2} \sqrt[3]{\frac{J_x}{6\rho \nu^2}} = \frac{1}{2} \sqrt[3]{\frac{\mathrm{Re}_x}{6x}}$, we finally obtain

$$\mathrm{Nu}_x = \frac{C_0}{2\sqrt[3]{6}} (\mathrm{Re}_x)^{1/3}.$$

The mean value of the Nusselt number

$$\overline{\mathrm{Nu}} = \frac{1}{x} \int_0^x \mathrm{Nu}_x \, dx = \frac{3}{4} \mathrm{Nu}_x.$$

Let us now consider the dynamic problem and determine the law of variation of mass flow in a plane jet (which is the same for the two variants of temperature boundary conditions)

$$G = \int_{-\infty}^{+\infty} \rho u \, dy = \rho \frac{A}{B} x^{1/3} [F(+\infty) - F(-\infty)] = \sqrt[3]{36\rho^2 \nu J_x} \cdot x^{1/3}.$$

Let us also determine the law of variation (decrease) of the kinetic energy flux

$$E = \frac{1}{2}\int_{-\infty}^{+\infty} \rho u^3\, dy = \frac{J_x}{30\mu}\sqrt[3]{6\rho\nu^2 J_x^2}\cdot x^{-1/3}.$$

At the end of this section devoted to the plane laminar jet we consider the transformation of the variables as suggested by R. Mises [133, 212].

We introduce as independent variables for Eqs. (4.45) the coordinate $\xi = x$ and the stream function ψ with the well-known definition equations

$$u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x}.$$

With the substitution of variables $\frac{\partial}{\partial x} = \frac{\partial}{\partial\xi} - v\frac{\partial}{\partial\psi}$, $\frac{\partial}{\partial y} = u\frac{\partial}{\partial\psi}$, we can rewrite Eq. (4.45) in the form of the equation as given by R. Mises

$$\frac{\partial u^2}{\partial\xi} = \nu u\frac{\partial^2 u^2}{\partial\psi^2}. \tag{4.61}$$

Thi^ is a nonlinear equation of the type of the heat-conduction equation whose coefficient depends on the sought function. A general method of integration of this equation is not known. Approximation methods of integration are therefore of some interest. In particular, a method of successive approximations was given by L.G. Loytsyanskiy [133]. In the zeroth approximation of this method, the variable quantity u, the "coefficient" of the right-hand side of Eq. (4.61), is replaced by a characteristic constant. The solution of the linear equation obtained with the relevant boundary conditions is used in the iteration process carried out to obtain the higher approximations. This method may be of considerable interest since the solution of a heat-conduction-type equation (e.g., by the source method) enables us to take detailed initial conditions into account (the velocity distri-

bution at the orifice, etc.). As a rule, the zeroth approximation is, however, far from exact and the calculation of the following approximations is complicated and cumbersome. Owing to this the method could not achieve any remarkable spread in the theory of plane jets.

The integration of R. Mises' nonlinear equation by the method of finite differences must be considered more promising, in particular if static electro-integrators (SEI) are used for this purpose [142].

4.4. THE FAN-TYPE SOURCE JET

In this section we shall consider a laminar flow produced by a fan-type source jet. Let us assume that at all points of the slit issuing the jet the velocity vector lies in one and the same plane, i.e., in the origin cross section the projection of the velocity vector to the axis Oy of symmetry is identically equal to zero. In the absence of a "twist" the velocity in the origin cross section is directed along the radius, in the presence of it, it makes a certain angle with the radius in the plane $y = 0$. In the first case the jet is sometimes called a radial-slit jet; for generality we shall speak of a fan-type jet which may be twisted or nontwisted.

Just as in other cases of source jets we shall consider the motion at a sufficient distance from the orifice. We shall first consider the dymamic problem [132, 161] taking into account (analogously as in the plane jet) that the temperature boundary conditions are different.

Note that as to its "geometry" the fan-type jet in some way takes an intermediate position between a plane and an axisymmetric jet. The flow produced by a fan-type source jet is as in the plane jet symmetrical with respect to the plane $y = 0$ and, as in a round jet, it has axial symmetry. But the axis of symmetry is here perpendicular to the velocity vector, identical with the Oy axis, and not with the longitu-

dinal Ox axis as in the above problem on the axisymmetric jet. The fan-type jet is related with the latter by the continuity equation and the law of growth of the jet's cross-sectional area connected with this equation.

The equations of motion and continuity in the approximation of the boundary-layer theory, using the above denotations, can be written in the following form:

$$\left.\begin{aligned} &u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-\frac{w^2}{x}=\nu\frac{\partial^2 u}{\partial y^2},\\ &\frac{\partial p}{\partial y}=0,\\ &u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+\frac{uw}{x}=\nu\frac{\partial^2 w}{\partial y^2},\\ &\frac{\partial}{\partial x}(xu)+\frac{\partial}{\partial y}(xv)=0.\end{aligned}\right\} \qquad (4.62)$$

The boundary conditions for the system (4.62) read

$$\left.\begin{aligned} &v=0, \quad \frac{\partial u}{\partial y}=\frac{\partial w}{\partial y}=0 \quad \text{if} \quad y=0,\\ &u=w=0 \quad \text{if} \quad y=\pm\infty.\end{aligned}\right\} \qquad (4.63)$$

For unambiguousness of the solution of the problem, the boundary conditions (4.63) must be completed by integral conditions.

We multiply the first equation of System (4.62) by x and the fourth by u and add them:

$$\frac{\partial}{\partial x}(xu^2)+\frac{\partial}{\partial y}(xuv)-w^2=\nu\frac{\partial^2}{\partial y^2}(xu).$$

We integrate the equation obtained with respect to y from $-\infty$ to $+\infty$; taking the boundary conditions (4.63) into account, we then obtain

$$\frac{d}{dx}\int_{-\infty}^{+\infty} xu^2\,dy=\int_{-\infty}^{+\infty} w^2\,dy. \qquad (4.64)$$

From this equation it follows in particular that in the case of a nontwisted jet (if $w = 0$) the momentum flux (the "impulse" of the jet) remains constant along the jet:

$$J_x=2\pi x\int_{-\infty}^{+\infty}\rho u^2\,dy=\text{const.} \qquad (4.65)$$

In the case of a twisted fan-type jet we must use the strict condition (4.64) in order to solve the problem, for a slightly twisted jet, at a great distance from the source, Eq. (4.65) will, however, maintain its property as a limiting relation. The validity of the latter statement results in the fact that, as will be shown later, the laws of variation of the longitudinal (u) and the peripheral (w) velocity components with increasing distance to the source will be different:

$$u \sim \frac{1}{x}, \quad w \sim \frac{1}{x^2}.$$

While the term $w^2/x \sim 1/x^5$ in the first equation of the initial system, the other terms will be proportional to $1/x^3$. Consequently, at greater distances from the source, we may neglect the term w^2/x relative to the other terms of the first equation. Then, however, also in the integral relation (4.64) the right-hand side must be set equal to zero in this approximation.

In order to obtain a second integral condition we multiply the third equation of the system (4.62) by x^2 and add it to the fourth equation which has been multiplied preliminarily by xw. As a result we obtain

$$\frac{\partial}{\partial x}(x^2uw) + \frac{\partial}{\partial y}(x^2vw) = \nu \frac{\partial^2}{\partial y^2}(x^2w),$$

which we integrate from $y = -\infty$ to $y = +\infty$ across the boundary layer, the boundary conditions (4.63) being taken into account. This yields the following condition:

$$M_y = 2\pi x^2 \int_{-\infty}^{+\infty} \rho uw \, dy = \text{const} \qquad (4.66)$$

along the jet. Equation (4.66) expresses the law of conservation of angular momentum in the jet, relative to the axis of symmetry, Oy.

To solve our problem we put

$$u = Ax^{\alpha}F'(\varphi), \qquad w = Cx^{\varepsilon}\Phi(\varphi), \qquad \varphi = Byx^{\beta}. \tag{4.67}$$

Let us require that finite equations must not contain the coordinate x. For this purpose we put

$$\beta = \frac{\alpha - 1}{2} \tag{4.68}$$

and after a transformation we obtain

$$\begin{aligned} \frac{\nu B^2}{A} F''' &= \alpha F'^2 - \frac{3+\alpha}{2} FF'', \\ \frac{\nu B^2}{A} \Phi'' &= (1+\varepsilon) F'\Phi - \frac{3+\alpha}{2} F\Phi'. \end{aligned} \tag{4.69}$$

The constants of self-similarity, α, β and ε can be obtained from the integral conditions (4.65) and (4.66)

$$\beta = 2\alpha + 1, \qquad \varepsilon = \beta - \alpha - 2 \tag{4.70}$$

and, moreover,

$$\frac{A^2}{B} = \frac{J_x}{2\pi\rho J_1} \qquad \left(J_1 = \int_{-\infty}^{+\infty} F'^2 \, d\varphi\right), \tag{4.71}$$

$$C = \frac{B}{A} \frac{M_y}{2\pi\rho J_2} \qquad \left(J_2 = \int_{-\infty}^{+\infty} F'\Phi \, d\varphi\right). \tag{4.72}$$

When we solve Eqs. (4.68) and (4.70) simultaneously, we obtain the following values for the constants of self-similarity:

$$\alpha = \beta = -1, \qquad \varepsilon = -2. \tag{4.73}$$

Making use of the arbitrariness of the values of the constants A and B we can put in addition to the above relations

$$\frac{A}{B^2} = 2\nu; \tag{4.74}$$

such that the problem is reduced to integrating the system of ordinary differential equations

$$F''' + 2(FF')' = 0, \tag{4.75}$$

$$\Phi'' + 2(F\Phi)' = 0 \tag{4.76}$$

with the boundary conditions

$$\left.\begin{aligned} &F = 0, \qquad F' = \Phi = 1 \quad \text{if} \quad \varphi = 0, \\ &F' = \Phi = 0 \qquad\qquad\quad \text{if} \quad \varphi = \pm\infty, \end{aligned}\right\} \tag{4.77}$$

The equations (4.75) for F' and (4.76) for Φ and the boundary conditions are similar such that the functions F' and Φ may only differ by a constant factor which can be taken equal to unity since the velocity components u and w are determined with an accuracy that implies arbitrary values of the factors A and C. They will be determined in the following.

It should also be noted that Eq. (4.75) and the boundary conditions (4.77) for the determination of the function $F'(\varphi)$ are identical with Eq. (4.47) and the boundary conditions (4.48) to which we reduced the solution of the problem on the expansion of a plane laminar source jet. The solution of the present problem for the function $F'(\varphi)$ is therefore obtained immediately:

$$F = \operatorname{th}\varphi, \quad F' = \frac{1}{\operatorname{ch}^2\varphi} = 1 - \operatorname{th}^2\varphi, \quad F' = \Phi. \qquad (4.78)$$

To complete the solution of the problem we determine the values of the constants A, B and C from the conditions (4.71), (4.72) and (4.73):

$$A = \frac{1}{4}\sqrt[3]{\frac{9J_x^2}{2\pi^2\rho^2\nu}}, \quad B = \frac{1}{2}\sqrt[3]{\frac{3J_x}{4\pi\rho\nu^2}}, \quad C = \frac{3M_v}{8\pi\rho}\sqrt[3]{\frac{4\pi\rho}{3\nu J_x}}.$$

(Here we took into account that $J_1 = J_2 = \int\limits_{-\infty}^{+\infty} F'^2\, d\varphi = \frac{4}{3}$).

In this way we arrive at the final solution of the dynamic problem

$$\left.\begin{aligned} u &= \frac{1}{4}\sqrt[3]{\frac{9J_x^2}{2\pi^2\rho^2\nu}}\,\frac{1}{x}\,\frac{1}{\operatorname{ch}^2\varphi}, \\ v &= \frac{1}{2}\sqrt[3]{\frac{3\nu J_x}{4\pi\rho}}\,\frac{1}{x}\,\frac{2\varphi - \operatorname{sh}2\varphi}{\operatorname{ch}^2\varphi}, \\ w &= \frac{3}{8}M_v\sqrt[3]{\frac{4}{3\pi^2\rho^2\nu J_x}}\,\frac{1}{x^2}\,\frac{1}{\operatorname{ch}^2\varphi} \\ &\left(\varphi = \frac{y}{2x}\sqrt[3]{\frac{3J_x}{4\pi\rho\nu^2}}\right). \end{aligned}\right\} \qquad (4.79)$$

Let us now turn to the thermal problem and consider its solution (in a similar way as we treated the above case of the plane jet) for

two types of boundary conditions for the temperature:

a)
$$\left.\begin{array}{lll}\frac{\partial T}{\partial y}=0 & \text{if} & y=0, \\ T=T_\infty & \text{if} & y=\pm\infty\end{array}\right\} \tag{4.80a}$$

and

b)
$$\left.\begin{array}{lll}T=T_1 & \text{if} & y=+\infty, \\ T=T_2 & \text{if} & y=-\infty.\end{array}\right\} \tag{4.80b}$$

As in the case of the plane jet, in order to solve the energy equation

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=a\frac{\partial^2 T}{\partial y^2} \tag{4.81}$$

we introduce different self-similarity transformations for the boundary conditions (4.80a) and (4.80b):

a) for the "symmetrical" boundary conditions (4.80a)

$$\frac{T-T_\infty}{T_m-T_\infty}=\theta_a(\varphi), \qquad T_m-T_\infty=\Gamma x^\gamma; \tag{4.82a}$$

b) for the "asymmetric" boundary conditions (4.80b)

$$\frac{T-T_2}{T_1-T_2}=\theta_b(\varphi). \tag{4.82b}$$

In the first case, to obtain a complete solution of the problem we must add to the boundary conditions (4.80a) the integral condition of conservation of the surplus heat content in the jet:

$$2\pi x\int_{-\infty}^{+\infty}\rho c_p u(T-T_\infty)\,dy=Q=\text{const}, \tag{4.83}$$

derived analogously as the conditions (4.65) and (4.66).

Substituting the expressions for the longitudinal velocity component u from (4.79) and the temperature difference $T - T_\infty$ from (4.82a) in th integral condition (4.83) enables us to determine the constant of self-similarity γ and the constant Γ

$$\gamma=-1, \quad \Gamma=\frac{Q}{2\pi\rho c_p J_2}\sqrt[3]{\frac{4\pi\rho}{3\nu J_x}} \qquad \left(J_2=\int_{-\infty}^{+\infty}F'\theta_a\,d\varphi\right). \tag{4.84}$$

The heat transfer equation (4.81) and the boundary conditions (4.80a) transform with the value of $\gamma = -1$ obtained to an ordinary dif-

ferential equation

$$\theta_a'' + 2Pr\,(F\theta_a)' = 0 \tag{4.85}$$

with the boundary conditions

$$\left.\begin{array}{l} \theta_a' = 0 \ \text{ if } \ \varphi = 0, \\ \theta_a = 0 \ \text{ if } \ \varphi = \pm\infty. \end{array}\right\} \tag{4.86}$$

In the second case Eq. (4.81) and the boundary conditions (4.80b) can be rewritten in the form

$$\theta_b'' + 2\,Pr\,F\theta_b' = 0, \tag{4.87}$$

$$\left.\begin{array}{l} \theta_b = 1 \ \text{ if } \ \varphi = +\infty, \\ \theta_b = 0 \ \text{ if } \ \varphi = -\infty. \end{array}\right\} \tag{4.88}$$

The equations and boundary conditions (4.85) and (4.86) and also (4.87) and (4.88) coincide with the corresponding equations and boundary conditions (4.54), (4.55) and (4.58), (4.59) for the problem of the expansion of a plane jet; we can therefore rewrite immediately all the results for the temperature distribution in the fan-type jet.

In the first case (symmetrical thermal boundary layer)

$$\theta_a = \frac{1}{(\operatorname{ch}\varphi)^{2Pr}}, \quad \theta_a = (F')^{Pr}, \quad \frac{T-T_\infty}{T_m-T_\infty} = \left(\frac{u}{u_m}\right)^{Pr}, \tag{4.89}$$

$$T - T_\infty = \frac{Q}{J_3 c_p \sqrt[3]{6\pi^2\rho^2\nu J_x}}\,\frac{1}{x}\,\frac{1}{(\operatorname{ch}\varphi)^{2Pr}}$$

$$\left(J_3 = \int\limits_{-\infty}^{+\infty} (F')^{Pr+1}\,d\varphi = \begin{cases} \dfrac{\pi}{2} \ \text{ if } \ Pr = 0{,}5, \\ \dfrac{4}{3} \ \text{ if } \ Pr = 1{,}0, \\ \dfrac{16}{15} \ \text{ if } \ Pr = 2{,}0. \end{cases} \right)$$

In the second case ("asymmetric" thermal boundary layer)

$$\frac{T-T_2}{T_1-T_2} = \theta_b = C_0 \int\limits_{-\infty}^{\varphi} (\operatorname{ch}\varphi)^{-2Pr}\,d\varphi, \quad C_0 = \left[\int\limits_{-\infty}^{+\infty} (\operatorname{ch}\varphi)^{-2Pr}\,d\varphi\right]^{-1}. \tag{4.90}$$

The values of the constant C_0 were derived above for several Prandtl numbers.

The nondimensional temperature distributions in the fan-type and the plane free laminar jets are the same. The latter are shown in Fig. 4.4.

The coincidence of the self-similar equations and their solutions (i.e., of the universal nondimensional velocity and temperature distributions) for the dynamic and the thermal problems on the expansion of a plane and a fan-type free jet is a general property. It is also encountered in the self-similar solutions for half-limited laminar jets and in analogous problems of the theory of free and half-limited turbulent jets.

To obtain the total characteristic of the problem we also calculate the integral quantities of the fan-type jet.

The flow rate per second of the fluid in the jet is equal to

$$G = 2\pi x \int_{-\infty}^{+\infty} \rho u \, dy = 2\sqrt[3]{6\pi^2\rho^2\nu J_x}\, x \sim x.$$

The kinetic energy flux is

$$E = \pi x \int_{-\infty}^{+\infty} \rho u^3 \, dy = \frac{3J_x^{5/3}}{20\pi\mu} \sqrt[3]{\frac{4}{3}\pi\rho\nu^3}\, \frac{1}{x} \sim x^{-1}$$

and the local Nusselt number is obtained as

$$\mathrm{Nu}_x = \frac{x}{T_1 - T_2} \frac{\partial T}{\partial y}\Big|_{y=0} = \frac{\partial \theta_2}{\partial y}\Big|_{y=0} x = \theta_2'\big|_{\varphi=0}\, x \frac{\partial \varphi}{\partial y} = BC_0 = \mathrm{const}.$$

Besides the plane fan-type jet (where the initial velocity vector lies in the plane $y = 0$) the so-called conic radial-slit jet considered by L.G. Loytsyanskiy [132] and H. Squire [171] is of interest. Such a jet which flows out of an annular slit produces in the initial cross section a divergent cone with a vertex angle of 2α between the limits $\alpha = \pi/2$ (plane fan-type jet) and $\alpha = 0$ (efflux from a ring parallel to the axis Ox).

REFERENCES

25, 56, 86, 87, 122, 130, 131, 132, 133, 134, 135, 151, 152, 155, 157, 161, 174, 193, 197, 198, 212, 231, 243, 294, 304.

Manuscript Page No.	[Footnotes]
8	Note that in the extended scheme of a turbulent jet, suggested by G.N. Abramovich [1, 4] for the flow sections of the jet in the immediate proximity of the nozzle and away from it the terms "initial" and "basic" jet sections have been adopted.
12	We agree in calling a jet "submerged" if it issues into a space filled by a fluid of the same physical properties as the fluid of the jet.
19	Under these numbers the books or papers can be found in the Reference List at the end of the book.
22	A fan-type jet in which the initial momentum is directed radially in the plane of symmetry is considered separately.
37	At first sight it seems that the ratio M_x/J_x could serve as this quantity. It does not exist, however, for a nontwisted jet while in the presence of a "twist" it does not enter the expression for v_R in a second approximation.
43	If $v_\theta = 0$ the values of $\alpha = 1$ and $\delta = 2$ correspond to the motion of a viscous fluid in a wedge-type space formed by intersecting planes (Gamel's problem [122]).
45	The residual components of the heat flux density are obviously equal to zero. This results from the definition of the flux of an arbitrary scalar quantity through a spherical surface centered at the source of this quantity.
55	A solution of the analogous dynamic problem with boundary conditions corresponding to a viscous fluid ($v_R = 0$, $v_\theta = 0$ at the surface of the cone) was obtained in the form of series in a paper by N.A. Slezkin [172].
61	By virtue of the symmetry of the flow with respect to the plane $\theta = \pi/2$ the solution is developed for the upper semispace ($0 \leq \theta \leq \pi/2$). In order to obtain a solution for the lower semispace we must replace in Eq. (3.20) ω by $-\omega$ (also in the integral conditions of the problem).
74	This is connected with the fact that for a plane jet motion no self-similar solutions exist (see Chapter 2).
75	The coordinate x is directed along the initial momentum.
83	Note that the form of the expressions for the velocity, the temperature, etc., and also for the reduced coordinate φ written as power-function complexes predetermines for the self-similar solutions the constancy of the value of the in-

tegral taken over the jet cross section, of the form $\int u^{\sigma} ds = \text{const},$ where $\sigma = \frac{\beta + k}{\alpha}$ and $ds = (2\pi y)^k dy,$ (where k = 0, 1). For free jets σ = 2 and this integral condition coincides with the more general condition of conservation $J_x = \int \rho u^2 ds =$ = const, which is in no connection with an assumption of self-similarity of the flow. In those cases where the general integral invariant of the problem is unknown, an equation of the form of $\int u^{\sigma} ds = \text{const} \neq 0$ under zero boundary conditions guarantees the possibility of obtaining a nontrivial solution.

84 Just as in the previous part we only consider the case of a "slight twist." In this case, although the pressure is maintained in the second equation of the system (4.4) it may be omitted in the first equation in connection with the sharp drop ($\sim x^{-4}$) with increasing distance to the source. In the case of a "strong twist" ($w \gg u$) this simplification is inadmissible since here, as shown by experiment, counterflows arise in the paraxial region of the flow.

91 Note that an analogous solution of the problem on the plane source jet leads to a considerably smaller difference of the constants: the constants A and A' differ only by 2% while the difference between the constants B and B' amounts to about 60%. The agreement can be improved by raising the degree of the polynomial (4.32).

94 With Pr < 1 we must have $\delta_T > \delta$. But since beyond a layer of the thickness δ the flow velocity is equal to zero and it is rather artificial to extend the method of solution to this range of values of Prandtl's number.

Chapter 5

THE JET IN THE CO-MOVING FLOW

5.1 THE METHOD OF SMALL PERTURBATIONS

The idea of the laminar source jet has its advantages and disadvantages just as any other schematic conception. An advantage is the possibility of passing over from partial differential equations to ordinary differential equations and the solution of the latter without the necessity of taking detailed initial conditions into account. These achievements were demonstrated by way of examples considered above. As regards the disadvantages of the schematic conception chosen (if this term may be applied to the fundamental, essential properties of source jets), the impossibility of obtaining a solution and the corresponding representation of the true nature of the motion for flows in a jet near the orifice of the nozzle may be considered as such. In a series of problems it is, however, of great interest to know, for example, the variation of the initial velocity distribution and the like.

To overcome these difficulties, i.e., to obtain a direct solution to problems of nonselfsimilar flows, we must have recourse, as already mentioned, to numerical calculations, e.g., by means of computers. Besides this, the approximation methods of calculation are of well-known interest as they permit a simple and illustrative interpretation.

One of these approximation methods applied in the theory of

jets and in many other fields of mathematical physics as well, is the method of linearization of the fundamental equations, usually called the method of small perturbat-ons. If applied to calculating laminar (and also turbulent) jets, the method of small perturbations results in a transition from the nonlinear boundary layer equations to a well-known linear equation of the type of the heat-conduction equation. As already mentioned, the latter is relatively easy to solve both analytically and with the help of various integrators.

Note that the method of small perturbations may also be applied to self-similar flows produced by source-jets since the motion far away from the source of perturbations in the same sense belongs to the class of flows with small perturbations. Such examples will be dealt with below.

The main significance of the method of small perturbations consists however, in the possibility of applying it to problems which cannot be reduced to self-similar ones.

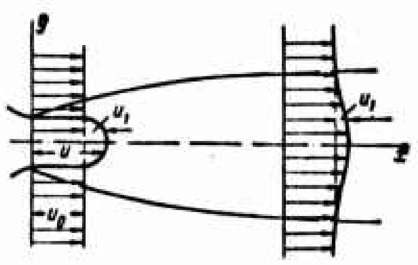

Fig. 5.1 Schematic representation of the expansion of a jet in a co-moving flow.

In order to illustrate the method as a whole, we consider the expansion of a jet of finite dimensions in an infinite, co-moving uniform flow. A single limitation (which, generally speaking, is essential) imposed on the flow (see schematic representation of Fig.5.1 consists in the supposition that the quantity of the "excessive velocity" $u_1 = u - u_\infty$ is at all points (and, first of all, for the maximum value u_{1m})essentially smaller than the velocity u_∞ of the co-moving flow. Thus we have $u_{1m} << u_\infty$.

In the approximation of the theory of the laminar boundary layer where the transverse velocity component is considerably smaller than

the longitudinal component ($v \ll u \approx u_\infty$), it is obvious that we must assume the order of the quantity v to be the same as that of u_1. In other words, $v \ll u_\infty$.

On this basis, we may rewrite the boundary layer equations for a plane and nontwisted axisymmetric flow

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{1}{y^k} \frac{\partial}{\partial y}\left(y^k \frac{\partial u}{\partial y}\right), \quad \frac{\partial}{\partial x}(y^k u) + \frac{\partial}{\partial y}(y^k v) = 0 \qquad (5.1)$$

in a form corresponding to the method of small perturbations. For this purpose, we replace in these equations the velocity u by the sum $u_\infty + u_1$ and then omit all terms which are small of second order in u_1. As the result of this linearization, we obtain the quantity

$$u_\infty \frac{\partial u_1}{\partial x} = \frac{\partial u_1}{\partial \tau} = \nu \frac{1}{y^k} \frac{\partial}{\partial y}\left(y^k \frac{\partial u_1}{\partial y}\right), \qquad (5.2)$$

where $\tau = \frac{x}{u_\infty}$ — is the "time" of propagation of perturbation, transferred in the linearized flow with the constant velocity u_∞. In the same way, we came to the well-known unidimensional heat-conduction (diffusion) equation which must be integrated instead of the system of equations (5.1) under the corresponding boundary conditions for the "velocity surplus" u_1. The latter can be written in the following form:

$u_1 = u_{10}(y)$ within the limits of the outlet cross section of the nozzle } with x = 0

$u_1 = 0$ beyond the nozzle } with x = 0

and

$$u_1 = 0, \ \frac{\partial u_1}{\partial y} = 0 \quad \text{if} \quad x > 0 \text{ и } y \to \infty.$$

The solution obtained in nondimensional form can be represented by the dependence on the relative coordinates $\frac{x}{r_0}, \frac{y}{r_0}$, the parameter $m = \frac{u_{10m}}{u_\infty}$ and Reynolds number $Re_0 = \frac{u_{10m} r_0}{\nu}$, where r_0 and u_{10m} are the characteristic values of nozzle dimensions and efflux velocity ex-

cessive relative to the velocity of the co-moving flow, u_∞.

It must be mentioned that a solution by means of the method of small perturbations does not at all require a restriction to plane or axisymmetric problems as this was necessary with Eq. (5.1). For the efflux of a jet from a square, rectangular or any other such form of nozzle, i.e., for a three-dimensional (spatial) jet flow, the linearization of the equation has the form of a two-dimensional equation of the type of the heat-conduction equation $(\tau = \frac{x}{u_\infty})$

$$\frac{\partial u_1}{\partial \tau} = \nu\left(\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2}\right),$$

which is also easy to solve, e.g., by the method of sources or other methods.

If in solutions obtained by the method of small perturbations, we consider the limiting relations for a great distance from the nozzle (with $\frac{x}{r_0} \gg 1$), these will correspond to a self-similar flow.

The constant of self-similarity, β, entering a power function of self-similarity of the form $\varphi = Byx^\beta$, can be found by substituting the expressions for $u_1, \frac{\partial u_1}{\partial x}$ etc., in the basic equations. The homogeneity of the linear equation (5.2) will here permit a definition of the constant β independently of the constant α entering the expression $u_1 = Ax^\alpha \frac{F'}{\varphi^k}$. The latter in its turn is found from the integral condition.

The simplest considerations of dimensionality, however, enable us to estimate immediately the value of the constant β. For example, in the case of a plane (or axisymmetric) jet, it follows from the equation

$$\frac{\partial u_1}{\partial \tau} = \nu \frac{1}{y^k} \frac{\partial}{\partial y}\left(y^k \frac{\partial u_1}{\partial y}\right)$$

that the nondimensional argument for a self-similar solution can only be represented in the form of the function φ = const $y/(\nu x/u_\infty)^{-2}$, β = −1/2, since at large distances from the nozzle the initial para-

meters r_0 and u_{10m} do not influence the motion.

Note finally that the method of small perturbations can also be a-plied to slightly twisted jets, i.e., to the peripheral velocity w under the supposition of the motion's self-similarity. As to the thermal problem, it can be solved even without using the method of small perturbations (by virtue of the linearity of the heat transfer equation) after the determination of the perturbed motion. This, however, is a very cumbersome way. It is therefore expedient to solve not only the dynamic problem but also the thermal problem by the same method of small perturbations. The corresponding examples will be given in the following section.

5.2 EXAMPLES OF SOLUTIONS BY THE METHOD OF SMALL PERTURBATIONS

Let us now use the general considerations developed in the preceding section on the application of the method of small perturbations in the theory of jet flows in order to solve concrete problems.

Let us first consider the problem of a plane jet issuing from a slot nozzle of the finite length of $2l$ and expanding in a co-moving uniform flow of fluid of the same physical properties, which streams with the velocity u_∞. In accordance with what has been said above, the velocity and the temperature of the jet at the nozzle orifice are assumed to differ only slightly from the velocity and the temperature of the co-moving flow (see Fig. 5.1).

As shown above, the solution of the dynamic and the thermal problems can be reduced under these conditions to an integration of equations of the type of the heat-conduction equation,

$$\left.\begin{aligned}\frac{\partial u_1}{\partial \tau} &= \nu\frac{\partial^2 u_1}{\partial y^2},\\ \frac{\partial T_1}{\partial \tau} &= a\frac{\partial^2 T_1}{\partial y^2}\end{aligned}\quad\left(\tau = \frac{x}{u_\infty}\right)\right\} \tag{5.3}$$

for the excessive velocity $(u_1 = u - u_\infty)$ and the temperature $(T_1 = T - T_\infty)$.

The solution of Eq. (5.3) must satisfy the following initial and boundary conditions:

$$\left.\begin{array}{llll} u_1 = u_{10}(y), & T_1 = T_{10}(y) & \text{if} & -l<y<l, \\ u_1 = 0, & T_1 = 0 & \text{if} & |y|>l, \end{array}\right\} x = 0, \qquad \left.\begin{array}{llll} \dfrac{\partial u_1}{\partial y} = 0, & \dfrac{\partial T_1}{\partial y} = 0 & \text{if} & y=0, \\ u_1 = 0, & T_1 = 0 & \text{if} & y = \pm\infty, \end{array}\right\} x>0. \tag{5.4}$$

Note that in contrast to jet flows which do not satisfy the requirements of the method of small perturbations, Eq. (5.3) for the temperature can be integrated independently of the velocity equation. To begin with, let us turn to the solution of the dynamic problem. As regards the thermal problem, its solution may be obtained by way of a simple replacement of u_1 by T_1 (u_{10} by T_{10}) in the solution of the dynamic problem; the kinematic coefficient of viscosity, ν, is here replaced by the heat-conduction coefficient α. This replacement refers to the symmetrical jet at whose boundaries both the velocity values and the temperature values are the same. But the linearization of the equations to such of the type of the heat-conduction equation enables us in each problem (the dynamic and the thermal problems or in both simultaneously) to obtain a solution even with different boundary values at $y = +\infty$ and $y = -\infty$.

The solution of an equation of the type of (5.3) with the boundary conditions (5.4) is well known [188]:

$$u_1(x, y) = \frac{1}{2\sqrt{\pi\nu\tau}} \int_{-l}^{+l} u_{10}(\xi) \exp\left[-\frac{(\xi-y)^2}{4\nu\tau}\right] d\xi. \tag{5.5}$$

In the particular case where the velocity in the jet at the outlet of the slit is constant $u_{10}(\xi) = u_{10} = \text{const}$, we shall have

$$\frac{u_1}{u_{10}} = \frac{1}{2\sqrt{\pi\nu\tau}} \int_{-l}^{+l} \exp\left[-\frac{(\xi-y)^2}{4\nu\tau}\right] d\xi. \tag{5.6}$$

The latter expression can be rewritten in a form more convenient for application if we substitute the variables as follows:

$$\frac{\xi - y}{2\sqrt{\nu\tau}} = \eta, \quad d\xi = 2\sqrt{\nu\tau}\, d\eta.$$

After this transition to new variables, we obtain

$$\frac{u_1}{u_{10}} = \frac{1}{\sqrt{\pi}} \int_{-\frac{l+y}{2\sqrt{\nu\tau}}}^{\frac{l-y}{2\sqrt{\nu\tau}}} \exp(-\eta^2)\, d\eta$$

or, having in mind that the integrand is an even function,

$$\frac{u_1}{u_{10}} = \frac{1}{2}\left\{\operatorname{erf}\left(\frac{l+y}{2\sqrt{\nu\tau}}\right) + \operatorname{erf}\left(\frac{l-y}{2\sqrt{\nu\tau}}\right)\right\}. \qquad (5.7)$$

(In this equation $\operatorname{erf} z = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-\alpha^2)\, d\alpha$ is an error integral).

Let us rewrite Eq. (5.7) in nondimensional variables

$$\frac{u_1}{u_{10}} = \frac{1}{2}\left\{\operatorname{erf}\left(\frac{1+\bar{y}}{\sqrt{\frac{m\bar{x}}{\mathrm{Re}_0}}}\right) + \operatorname{erf}\left(\frac{1-\bar{y}}{\sqrt{\frac{m\bar{x}}{\mathrm{Re}_0}}}\right)\right\}, \qquad (5.8)$$

where

$$\bar{y} = \frac{y}{l}, \quad \bar{x} = \frac{x}{l}, \quad \mathrm{Re}_0 = \frac{l u_{10}}{\nu}, \quad m = \frac{u_{10}}{u_\infty}.$$

Consequently, the velocity in a jet of finite dimensions is determined by the expression

$$\frac{u}{u_\infty} = 1 + \frac{m}{2}\left\{\operatorname{erf}\left(\frac{1+\bar{y}}{2}\sqrt{\frac{\mathrm{Re}_0}{m\bar{x}}}\right) + \operatorname{erf}\left(\frac{1-\bar{y}}{2}\sqrt{\frac{\mathrm{Re}_0}{m\bar{x}}}\right)\right\}. \qquad (5.9)$$

From Eq. (5.9), it is easy to obtain in particular the law of velocity variation along the axis of the jet $(\bar{y} = 0)$:

$$\frac{u_{\max}}{u_\infty} = 1 + m \operatorname{erf}\left(\frac{1}{2}\sqrt{\frac{\mathrm{Re}_0}{m\bar{x}}}\right). \qquad (5.10)$$

According to the above, we can write the solution of the thermal problem in the form

$$T_1(x, y) = \frac{1}{2}\sqrt{\frac{\mathrm{Pr}}{\pi\nu\tau}} \int_{-l}^{+l} T_{10}(\xi) \exp\left[-\frac{(\xi - y)^2}{4\nu\tau}\mathrm{Pr}\right] d\xi. \qquad (5.11)$$

For a uniform temperature distribution at the nozzle outlet $[T_{10}(\xi) = T_{10} = \mathrm{const}]$ we obtain

$$\frac{T_1}{T_{10}} = \frac{1}{2}\left\{\operatorname{erf}\left(\frac{1+\bar{y}}{2}\sqrt{\frac{\mathrm{Re}_0 \mathrm{Pr}}{m\bar{x}}}\right) + \operatorname{erf}\left(\frac{1-\bar{y}}{2}\sqrt{\frac{\mathrm{Re}_0 \mathrm{Pr}}{m\bar{x}}}\right)\right\}. \qquad (5.12)$$

The temperature distribution in a plane jet of finite dimensions is thus determined by the expression

$$\frac{T}{T_\infty} = 1 + \frac{m_T}{2}\left\{\operatorname{erf}\left(\frac{1+\bar{y}}{2}\sqrt{\frac{\mathrm{Re}_0\,\mathrm{Pr}}{m\bar{x}}}\right) + \operatorname{erf}\left(\frac{1-\bar{y}}{2}\sqrt{\frac{\mathrm{Re}_0\,\mathrm{Pr}}{m\bar{x}}}\right)\right\}. \quad (5.13)$$

(Here we introduced the additional denotation $m_T = \frac{T_{10}}{T_\infty}$.)

Along the axis of the jet ($\bar{y} = 0$) the temperature varies according to the following law:

$$\frac{T_{max}}{T_\infty} = 1 + m_T \operatorname{erf} \frac{1}{2}\sqrt{\frac{\mathrm{Re}_0\,\mathrm{Pr}}{m\bar{x}}}. \quad (5.14)$$

Note that also in this nonselfsimilar flow one of the fundamental properties of jet motions is con erved, namely the similarity of the temperature and velocity distributions, under similar boundary conditions and a Prandtl number equal to unity.

In a quite analogous way, we can investigate the laws governing the expansion of an axisymmetric jet issuing from a round opening into a uniform wake. Under the same suppositions as in the problem of the plane jet of finite dimensions ($u_1 = u - u_\infty \ll u_\infty$; $T_1 = T - T_\infty \ll T_\infty$), we start from the equations

$$\begin{aligned} \frac{\partial u_1}{\partial \tau} &= \frac{\nu}{y}\frac{\partial}{\partial y}\left(y\frac{\partial u_1}{\partial y}\right), \\ \frac{\partial T_1}{\partial \tau} &= \frac{a}{y}\frac{\partial}{\partial y}\left(y\frac{\partial T_1}{\partial y}\right) \end{aligned} \quad (5.15)$$

with the initial and boundary conditions

$$\left.\begin{array}{llll} \left.\begin{array}{lll} u_1 = u_1(y), & T_1 = T_1(y) & \text{if } 0<y<l, \\ u_1 = 0, & T_1 = 0 & \text{if } y>l, \end{array}\right\} x = 0, \\ \left.\begin{array}{ll} \frac{\partial u_1}{\partial y} = \frac{\partial T_1}{\partial y} = 0 & \text{if } y = 0, \\ u_1 = 0,\ T_1 = 0,\ \frac{\partial u_1}{\partial y} = \frac{\partial T_1}{\partial y} = 0 & \text{if } y = \infty, \end{array}\right\} x > 0. \end{array}\right\} \quad (5.16)$$

The solution of Eqs. (5.15) under the conditions (5.16) and uniform velocity and temperature distributions in the nozzle orifice is also well known:

$$\left.\begin{aligned} \frac{u_1}{u_{10}} &= \int_0^\infty I_0(\bar{y}\lambda) I_1(\lambda) \exp\left[-\frac{m\bar{x}}{\mathrm{Re}_0}\lambda^2\right] d\lambda, \\ \frac{T_1}{T_{10}} &= \int_0^\infty I_0(\lambda\bar{y}) I_1(\lambda) \exp\left[-\frac{m\bar{x}}{\mathrm{Re}_0 Pr}\lambda^2\right] d\lambda. \end{aligned}\right\} \qquad (5.17)$$

In the latter equations I_0 and I_1 are Bessel functions of first kind and of zeroth and first order, respectively, the other designations are the same as in the problem of the plane jet*.

The assumption of a similarity of the velocity and temperature distributions with Pr = 1 in this case remains in force.

The above problems on jets of finite dimensions referred to the motion of an incompressible fluid. The solutions obtained describe (within the framework of the method of small perturbations), however, also the flow of a compressible gas. In this case, ν and α are the respective values in the nonperturbed wake [25].

As regards the problems of the expansion of jets of compressible gas discharged from nozzles of finite dimensions, they can be solved more strictly either by numerical integration of the corresponding equations or by the method of successive approximations. A series of solutions of this kind were given by Bay Shi-i [25].

Let us now consider the limiting transition to the case of the expansion of a source jet in a uniform wake, by way of the example of a plane jet of finite dimensions. For this purpose, we return to Eq. (5.6) and, considering the width $2l$ of the nozzle sufficiently small, we apply the theorem of the mean to the integral in the right-hand side. We obtain as a result

$$u_1 = \frac{u_{10} 2l}{2\sqrt{\pi\nu\tau}} \exp\left[-\frac{(\xi_0 - y)^2}{4\nu\tau}\right] \quad (-l < \xi_0 < l).$$

The momentum of the jet in the outlet cross section is equal to

$$J_x = \rho u_{10} u_\infty 2l,$$

disregarding the second-order small quantities; hence follows that $u_{10}2l = \frac{J_x}{\rho u_\infty}$, so that we obtain for u_1

$$u_1 = \frac{J_x}{2\rho u_\infty \sqrt{\pi \nu \tau}} \exp\left[-\frac{(\xi_0 - y)^2}{4\nu\tau}\right].$$

In the limiting transition with $l \to 0$ we shall suppose that at the same time $u_{10} \to \infty$, while the momentum J_x remains constant, equal to

$$J_x = \int_{-\infty}^{+\infty} \rho u_\infty u_1 \, dy.$$

Under these suppositions, we shall have finally

$$u_1 = \frac{J_x}{\rho \sqrt{4\pi \nu u_\infty x}} \exp(-\varphi^2) \quad \left(\varphi^2 = \frac{u_\infty}{4\nu} \frac{y^2}{x}\right). \tag{5.18}$$

An analogous limiting transition carried out in the righthand side of Eq. (5.11), under the condition that in the jet of the excess flow the following heat content is retained:

$$Q = \int_{-\infty}^{+\infty} \rho c_p u_\infty T_1 \, dy = \text{const}$$

results in the following expression for the excessive temperature:

$$T_1 = \frac{Q}{\rho c_p} \sqrt{\frac{Pr}{4\pi \nu u_\infty x}} \exp(-Pr\,\varphi^2). \tag{5.19}$$

Using the method of small perturbations the same results can be obtained for the excessive velocity u_1 and the temperature T_1 by way of an immediate integration of the initial system of equations of expansion of a plane source jet in a uniform wake under corresponding boundary conditions and integral conditions. A solution by the method of small perturbations may be obtained in a very simple and clear way with the help of dynamic or static hydro- or electrointegrators [34, 35, 114, 142].

5.3 THE MIXING OF UNIFORM FLOWS

The problem of a plane boundary layer in the interface of two parallel flows has already been studied in a series of investigations.

W. Tollmien's paper [317] is devoted to the turbulent mixing in the interface of a semi-infinite uniform flow of incompressible fluid and a medium at rest, possessing the same physical properties. Later on this problem was generalized to the case of a motion in the same direction of two parallel flows, in the papers by A. Kuethe [258] and H. Görtler [250].

In the present section, we discuss the results of investigations of both the parallel motion of two uniform, plane-parallel laminar flows and the well-known schematic representation of counterflows (Fig. 0.4), i.e., the mixing zone produced by two uniform antiparallel flows.

Such a motion (for an axisymmetric turbulent jet) was considered in paper [114] in connection with the problem of jet stabilization of flames (see also [45]. Considering this problem in a first approximation, we adopt the usual assumptions of the boundary layer theory. In particular, we neglect for the mixing zone the pressure variations transverse to the flow due to the curvature of the streamlines of the fluid.

On the basis of these considerations, we shall now turn to the statement and solution of the problem on the mixing of uniform parallel or antiparallel flows of a viscous incompressible flow.

We shall assume that two flows of one and the same fluid move with the different velocities u_1 and u_2 and the temperatures T_1 and T_2 along a plate which is parallel to the Ox axis. The flows are assumed uniform, i.e., the initial velocity distribution produced by the flow around the plate is not taken into account; in other words, the "wall" parting the flow is identified with the zero streamline of each of the flows.

At the point $x = 0$, the end of the plate, the mixing of the

flows and the formation of a laminar boundary layer begin. The velocity and temperature distributions in this boundary layer are described by the solution of the following system of differential equations:

$$\left.\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}&=\nu\frac{\partial^2 u}{\partial y^2},\\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}&=0,\\ u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}&=a\frac{\partial^2 T}{\partial y^2},\end{aligned}\right\} \tag{5.20}$$

with the boundary conditions

$$\left.\begin{aligned} &\left.\begin{aligned} u=u_1,\quad v=0,\quad T=T_1 \quad &\text{if}\quad y>0,\\ u=u_2,\quad v=0,\quad T=T_2 \quad &\text{if}\quad y<0,\end{aligned}\right\} x<0,\\ &\left.\begin{aligned} u=u_1,\quad T=T_1 \quad &\text{if}\quad y=+\infty,\\ u=u_2,\quad T=T_2 \quad &\text{if}\quad y=-\infty,\end{aligned}\right\} x>0.\end{aligned}\right\} \tag{5.21}$$

Note that the value of the velocity u_2 may be either positive (parallel flows) or negative (antiparallel flows). In contrast to this, we always have $u_1 > 0$. The ratio u_2/u_1 is denoted by m; we have $-\infty<m<1$ * (the value $m = 0$ corresponds to the problem of the jet boundary).

Starting from physical considerations (constant values of velocity and temperature at the boundaries of the mixing zone) we shall seek a solution of the system of equations (5.20) under the boundary conditions (5.21) in the form

$$u=u_1F'(\varphi),\quad \frac{T-T_2}{T_1-T_2}=\theta(\varphi)\qquad (\varphi=Byx^{\beta}). \tag{5.22}$$

Substituting these functions and their derivatives in the equations of the system (5.20) and requiring that the variable x must not enter the transformed equations, we obtain

$$F'''+2FF''=0, \tag{5.23}$$

$$\theta''+2\mathrm{Pr}\,F\theta'=0 \tag{5.24}$$

and, in addition,

$$\beta=-\frac{1}{2},\quad B=\frac{1}{2}\sqrt{\frac{u_1}{\nu}}. \tag{5.25}$$

The boundary conditions (in the righthand semiplane) for the problem considered take the following form:

$$\left.\begin{array}{l} F' = 1, \quad \theta = 1 \text{ if } \varphi = +\infty, \\ F' = m, \quad \theta = 0 \text{ if } \varphi = -\infty. \end{array}\right\} \tag{5.26}$$

In order to determine the temperature distribution it is necessary to solve the dynamic problem beforehand. For this purpose, we use an iteration method (similar to that of [250]), representing the sought function $F(\varphi)$ in the form of a series

$$F(\varphi) = \sum_{i=0}^{\infty} (m-1)^i F_i(\varphi). \tag{5.27}$$

The first term of this series is put equal to the variable φ:

$$F_0(\varphi) = \varphi. \tag{5.28}$$

We substitute the function $F(\varphi)$, determined by the series (5.27) and its derivatives in Eq. (5.23) and equalize the coefficients in terms of the same powers of $(m - 1)$. This yields a system of differential equations for the determination of the function $F_i(\varphi)$:

$$\left.\begin{array}{l} F_1''' + 2\varphi F_1'' = 0, \\ F_2''' + 2\varphi F_2'' = -2F_1 F_1'', \\ \cdots\cdots\cdots\cdots\cdots \end{array}\right\} \tag{5.29}$$

which must satisfy the following boundary conditions:

$$\left.\begin{array}{l} F_1'(+\infty) = 0, \\ F_1'(-\infty) = 1, \\ F_i'(\pm\infty) = 0 \quad (i > 2). \end{array}\right\} \tag{5.30}$$

Integrating the first of Eqs. (5.29) with the boundary conditions (5.30) taken into account, we obtain

$$F_1(\varphi) = \frac{1}{2}\left\{\varphi - \int_0^{\varphi} (\operatorname{erf} s)\, ds + C_1\right\}. \tag{5.31}$$

Using the value of $F_1(\varphi)$, obtained, the second equation of the systems (5.29) can be used to determine the function $F_2(\varphi)$ and so on. But let us restrict ourselves to the first two terms of the series (5.27) as an estimation of the following terms carried out by H. Görtler [250] showed that they are relatively small.

In this case we have

$$F(\varphi)=\varphi+\frac{1}{2}(m-1)\left\{\varphi-\int_0^{\varphi}(\operatorname{erf} z)\,dz+C_1\right\}. \qquad (5.32)$$

The velocity components are determined by the expressions

$$\left.\begin{aligned} \frac{u}{u_1}&=F'(\varphi)=1+\frac{1}{2}(m-1)(1-\operatorname{erf}\varphi),\\ \frac{v}{u_1}\sqrt{Re}&=\varphi F'-F=\frac{1}{2}(m-1)\left\{\int_0^{\varphi}(\operatorname{erf} z)\,dz-\varphi\operatorname{erf}\varphi-C_1\right\}, \end{aligned}\right\} \qquad (5.33)$$

where $Re=\frac{u_1 x}{\nu}$.

The stream function can be represented in the form

$$\frac{\psi}{2\nu\sqrt{Re}}=\varphi+\frac{1}{2}(m-1)\left\{\varphi-\int_0^{\varphi}(\operatorname{erf} z)\,dz+C_1\right\}. \qquad (5.34)$$

Note that the first equation of (5.33) with $\varphi=0$ yields

$$u=\frac{u_1+u_2}{2}$$

in particular, with $m = 0$, we obtain $u=\frac{u_1}{2}$).

Such a symmetry of the distribution of the longitudinal velocity component relative to the Ox axis is rather unlikely. This has been proven by experimental data on turbulent motions ([4] and others).

It is a well-known property of the boundary layer equations [279] that besides the solution of the dynamic equations of system (5.20) in the form of a function $F(\varphi)$ there also exist other solutions which satisfy the same boundary conditions, in the form of $F(\varphi^*)$ where φ^* differs from φ by a certain constant.

The solution of the problem could be brought to an agreement with the experiment by choosing this constant, as it was done in paper [51] for a turbulent motion. For the laminar motion, however, no experimental data exist.

Note that the first equation of (5.33) may be rewritten in the form

$$\frac{u-u_2}{u_1-u_2}=\left(\frac{u}{u_1}\right)_{m=0}=\frac{1}{2}(1+\operatorname{erf}\varphi). \qquad (5.35)$$

The righthand side of Eq. (5.35) is universal, i.e., it is independent of m. It is essential that the expression $\frac{1}{2}(1+\operatorname{erf}\varphi)$ represents the value of $F(\varphi)$ for the case where $m = 0$.

Equation (5.35) shows that it would be possible to restrict oneself to a solution of the problem only for the case of $m = 0$ (boundary of the jet) since the velocity distribution in the presence of a parallel or an antiparallel flow

$$u = u_2 + (u_1 - u_2)\left(\frac{u}{u_1}\right)_{m=0} \qquad (5.36)$$

is entirely determined by the function

$$\frac{u}{u_1} = \frac{1}{2}(1+\operatorname{erf}\varphi).$$

It is obvious that this conclusion cannot be drawn *apriori*, it can only be obtained as a result of the solution (and is connected with the linearization of the problem in a first approximation).

Since the terms omitted of the series for $F(\varphi)$ are sufficiently small, the above statements can also be extended with some degree of accuracy to the total solution of (5.27).

This result (the possibility of a transition to the "excessive" motion) will of course also apply to the transverse velocity component:

$$\frac{v}{(u_1-u_2)\frac{1}{\sqrt{\mathrm{Re}}}} = \left(\frac{v}{u_1\frac{1}{\sqrt{\mathrm{Re}}}}\right)_{m=0} = \frac{1}{2}\left\{\varphi\operatorname{erf}\varphi - \int_0^\varphi(\operatorname{erf} z)\,dz + C_1\right\}$$

or

$$v = (u_1 - u_2)\left(\frac{v}{u_1}\right)_{m=0}.$$

When in Eq. (5.36) the longitudinal velocity component is expressed in terms of the stream function $(u = \frac{\partial\psi}{\partial y})$, we obtain a formula for the stream function of a complex motion in the form

$$\psi = \psi_2 + \frac{u_1-u_2}{u_1}\psi_{m=0}$$

or

$$\psi = u_2 y + (1-m)\psi_{m=0}. \qquad (5.37)$$

The latter formula which we obtained from the solution has a more general meaning: it expresses the law governing the addition of stream functions for the motion considered (the case where both flows are directed along the Ox axis), which is valid within the limits of the given approximation.

Let us now turn to the solution of the thermal problem. With the help of (5.23), we rewrite Eq. (5.24) in the form

$$\frac{\theta''}{\theta'} = \Pr\frac{F'''}{F''}.$$

Integrating this expression twice, we obtain

$$\theta(\varphi) = C_2\int_{\varphi}^{\infty}(F'')^{\Pr}\,d\varphi + C_3.$$

From the boundary conditions (5.26), we find

$$C_3 = 1,\quad C_2 = -\left[\int_{-\infty}^{+\infty}(F'')^{\Pr}\,d\varphi\right]^{-1}.$$

Consequently,

$$\theta(\varphi) = 1 - \int_{\varphi}^{\infty}(F'')^{\Pr}\,d\varphi\left[\int_{-\infty}^{+\infty}(F'')^{\Pr}\,d\varphi\right]^{-1}.$$

When we substitute the value of the function $F''(\varphi)$, calculated from Eq. (5.32)

$$F''(\varphi) = \frac{1-m}{\sqrt{\pi}}\exp(-\varphi^2),$$

in the latter expression we finally arrive at

$$\theta(\varphi) = \frac{1}{2}\{1 + \operatorname{erf}(\varphi\sqrt{\Pr})\}. \qquad (5.38)$$

The temperature distribution obtained is universal, i.e., it is independent of the parameter m. It also disposes of all the fundamental properties of temperature distributions in jet flows as already indicated previously (similarity of the velocity and temperature distributions with Pr = 1, etc.).

Here we restrict ourselves to these general considerations about

the properties of the solution obtained, postponing the detailed analysis of the flow pattern and the agreement of the solution with experimental data to Chapter 11 which deals with an investigation of an analogous problem for a turbulent flow.

Literature References

25, 45, 51, 114, 130, 133, 134, 151, 152, 174, 262, 264, 277. 279.

Manuscript Page No.	[Footnotes]
120	Construction of the profiles for an axisymmetric jet, particularly for nonuniform initial distributions $u_1(0,y)$ and $T_1(0,y)$, is made easier by referring to tables of the so-called P-functions [264].
123	Needless to say, the solutions obtained below also extend to the region of satellite flows for which $1 < m < \infty$, on introduction of the parameter $m' = 1/m < 1$ and reversal of the direction of the Oy-axis.

Chapter 6

SEMILIMITED JETS OF AN INCOMPRESSIBLE FLUID

6.1. THE PLANE JET

Among the various jet flows a jet expanding along solid surfaces is of considerable interest, especially for the turbulent motion. We shall agree in calling this kind of jet flows "along a wall" semilimited jets. These semilimited jets possess the peculiarity that the region of flow covered by them looks as if it were the result of a synthesis of two types of boundary layers, a free one and one along a wall. In connection with this, the laws governing the flow in a semilimited jet take an intermediate position between the laws effective in the boundary layer on a solid wall and those of a free jet-type boundary layer.

This circumstance has an influence on both the laws of increase of effective thickness of the jet, of decrease of maximum velocity, etc., and the integral invariants of the problem.

In this section we consider a plane laminar semilimited source jet of incompressible fluid moving along an infinitely large plate. Thereafter we shall deal with a generalization of this problem to other cases of motion (fan-type semilimited jet, jet along a cone) and also with the peculiarities of semilimited gas jets and finally, in the following part, with turbulent jets.

The flow in a plane laminar source jet which expands along a solid surface (see Fig. 0.5a) is described by a system of differential

boundary layer equations for an incompressible fluid:

$$\left.\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}, \\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \end{aligned}\right\} \tag{6.1}$$

with the boundary conditions

$$\left.\begin{aligned} u=v=0 \quad \text{if} \quad =0, \\ u=0 \quad \text{if} \quad =\infty. \end{aligned}\right\} \tag{6.2}$$

Let us first turn to the solution of the dynamic problem for a plane semilimited jet obtained by N.I. Akatnov [14]. As regards the thermal problem it is, because of the great difference of the temperature boundary conditions, expedient to consider it separately.

The self-similarity transformation

$$\frac{u}{u_m}=F'(\varphi), \quad u_m=Ax^{\alpha}, \quad \varphi=Byx^{\beta}, \tag{6.3}$$

often used above reduces the system of Eqs. (6.1) to one ordinary differential equation for the function $F(\varphi)$:

$$F'''+\frac{A}{2\nu B^2}[(\alpha+1)FF''-2\alpha F'^2]=0. \tag{6.4}$$

We have here the same relation linking the constants of self-similarity which was obtained previously when studying free jet flows:

$$\beta=\frac{\alpha-1}{2}. \tag{6.5}$$

The transverse velocity component entering Eq. (6.1) is determined by the expression

$$v=-\frac{A}{2B}x^{\frac{\alpha-1}{2}}[(\alpha-1)\varphi F'+(\alpha+1)F]. \tag{6.6}$$

As also in the case of free jets, the problem on the semilimited jet can be solved with the help of (besides the boundary conditions) an integral condition of conservation of a certain quantity, which is necessary in order to obtain a nontrivial solution and to de ermine the constants of self-similarity, α and β, and the constants A and B as well. Such a condition for a plane semilimited jet was found by

N.I. Akatnov [14].

Let us briefly derive this integral condition. We multiply the second equation of System (6.1) by u and add it to the first:

$$\frac{\partial}{\partial x}(u^2)+\frac{\partial}{\partial y}(uv)=\nu\frac{\partial^2 u}{\partial y^2}.$$

Integrating the latter equation across the boundary layer, taking the boundary conditions (6.2) into account, we obtain

$$\frac{\partial}{\partial x}\int_0^y u^2\,dy+uv=\nu\frac{\partial u}{\partial y}-\nu\left(\frac{\partial u}{\partial y}\right)_{y=0}, \tag{6.7}$$

hence we obtain in particular

$$\frac{d}{dx}\int_0^\infty u^2\,dy=-\nu\left(\frac{\partial u}{\partial y}\right)_{y=0}. \tag{6.8}$$

We multiply Eq. (6.7) by u and again integrate with respect to y between 0 and ∞:

$$\int_0^\infty u\frac{\partial}{\partial x}\left(\int_0^y u^2 dy\right)dy+\int_0^\infty u^2 v\,dy=-\nu\left(\frac{\partial u}{\partial y}\right)_{y=0}\int_0^\infty u\,dy. \tag{6.9}$$

The first integral in the lefthand side of the latter equation can be represented as the difference

$$\int_0^\infty u\frac{\partial}{\partial x}\left(\int_0^y u^2\,dy\right)dy=\frac{d}{dx}\int_0^\infty u\left(\int_0^y u^2\,dy\right)dy-\int_0^\infty\frac{\partial u}{\partial x}\left(\int_0^y u^2\,dy\right)dy.$$

The second integral of the righthand side can be transformed with the help of the continuity equation

$$\int_0^\infty\frac{\partial u}{\partial x}\left(\int_0^y u^2\,dy\right)dy=-\int_0^\infty\frac{\partial v}{\partial y}\left(\int_0^y u^2\,dy\right)dy,$$

which, after an integration by parts, yields

$$\int_0^\infty\frac{\partial u}{\partial x}\left(\int_0^y u^2\,dy\right)dy=-v_\infty\int_0^\infty u^2\,dy+\int_0^\infty u^2 v\,dy.$$

Returning to Eq. (6.9), we rewrite it in the form

$$\frac{d}{dx}\int_0^\infty u\left(\int_0^y u^2\,dy\right)dy+v_\infty\int_0^\infty u^2\,dy=-\nu\left(\frac{\partial u}{\partial y}\right)_{y=0}\int_0^\infty u\,dy. \tag{6.10}$$

The continuity equation of the transverse velocity component

yields

$$v_\infty = -\frac{d}{dx}\int_0^\infty u\,dy;$$

taking this and Eq. (6.8) into account, Eq. (6.10) can be represented in the form

$$\frac{d}{dx}\int_0^\infty u\left(\int_0^y u^2\,dy\right)dy - \left(\frac{d}{dx}\int_0^\infty u\,dy\right)\left(\int_0^\infty u^2\,dy\right) = \left(\frac{d}{dx}\int_0^\infty u^2\,dy\right)\left(\int_0^\infty u\,dy\right)$$

or

$$\frac{d}{dx}\left\{\int_0^\infty u\left(\int_0^y u^2\,dy\right)dy - \left(\int_0^\infty u\,dy\right)\left(\int_0^\infty u^2\,dy\right)\right\} = 0.$$

When the first integral in the last equation is integrated by parts, we obtain

$$\frac{d}{dx}\int_0^\infty u^2\left(\int_0^y u\,dy\right)dy = 0.$$

Consequently

$$\int_0^\infty \rho u^2\left(\int_0^y \rho u\,dy\right)dy = K = \text{const.} \qquad (6.11)$$

Instead of the integral invariant K = const we can now introduce a simpler one in the form

$$P = \int_0^\infty \rho u^\sigma\,dy = \text{const},$$

where $\sigma = \beta/\alpha$. In the problem given, with $\alpha = -\frac{1}{2}$, $\beta = -\frac{3}{4}$ (see below), it is obvious that $\sigma = 3/2$. Note that for self-similar semilimited jets an equation of the form*

$$P = \int_0^\infty \rho u^{\beta/\alpha}\,ds = \text{const},$$

will always exist; $ds = (2\pi y)^k dy$ for a plane ($k = 0$) and an xisymmetric ($k = 1$) jet, or $ds = 2\pi x\,dy$ for a fan-type jet and the like.

Let us substitute in Eq. (6.11) the expression for the longitudinal velocity component u from (6.3). We then obtain from the trans-

formed equation (6.11) of the form

$$\frac{A^3}{B^3} x^{3\alpha-2\beta} \int_0^\infty FF'^2 d\varphi = \frac{K}{\rho^3} = \text{const}$$

another relationship for the constants of self-similarity

$$\beta = \frac{3}{2}\alpha \tag{6.12}$$

and, moreover,

$$\frac{A^3}{B^3} = \frac{K}{\rho^3 J_1} \qquad \left(J_1 = \int_0^\infty FF'^2 d\varphi\right). \tag{6.13}$$

From Eqs. (6.5) and (6.12) we find

$$\alpha = -\frac{1}{2}, \qquad \beta = -\frac{3}{4}. \tag{6.14}$$

Taking the value obtained for α into account and assuming (in connection with the arbitrariness of the constants A and B) that

$$\frac{A}{B^3} = 4\nu, \tag{6.15}$$

we obtain the following differential equation instead of (6.4) for the function $F(\varphi)$ determining the velocity distribution:

$$F''' + FF'' + 2F'^2 = 0 \tag{6.16}$$

with the boundary conditions

$$F = F' = 0 \text{ if } \varphi = 0, \quad F' = 0 \text{ if } \varphi = \infty. \tag{6.17}$$

We multiply Eq. (6.16) by F and integrate once:

$$FF'' + F^2F' - \frac{1}{2}F'^2 = C_1, \tag{6.16a}$$

where it follows from the boundary conditions (6.17) that $C_1 = 0$. Multiplying the latter equation by $F^{-3/2}$ and integrating once more we obtain

$$F^{-1/2}F' + \frac{2}{3}F^{3/2} = C_2. \tag{6.16b}$$

From the condition for $\varphi = \infty$ we find the value of $C_1 = \frac{2}{3}F_\infty^{3/2}$. Consequently

$$F^{-1/2}F' + \frac{2}{3}(F^{3/2} - F_\infty^{3/2}) = 0. \tag{6.18}$$

When instead of $F(\varphi)$ the new function

$$f(\varphi) = \sqrt{F(\varphi)}, \qquad (6.19)$$

is introduced, we obtain the equation

$$f' + \frac{1}{3}(f^3 - f_\infty^3) = 0,$$

which is easy to solve in the form of

$$C_3 + \varphi = \frac{1}{2f_\infty^2} \ln \frac{f^2 + f f_\infty + f_\infty^2}{(f_\infty - f)^2} + \frac{\sqrt{3}}{f_\infty^2} \operatorname{arctg} \frac{2f + f_\infty}{f_\infty \sqrt{3}}.$$

The constant of integration C_3 is determined from the boundary conditions for $\varphi = 0$:

$$C_3 = \frac{\sqrt{3}}{f_\infty^2} \operatorname{arctg} \frac{1}{\sqrt{3}}.$$

Returning to the function $F(\varphi)$, we finally obtain

$$\varphi = \frac{1}{2F_\infty} \ln \frac{F + \sqrt{F F_\infty} + F_\infty}{(\sqrt{F_\infty} - \sqrt{F})^2} + \frac{\sqrt{3}}{F_\infty}\left\{\operatorname{arctg} \frac{2\sqrt{F} + \sqrt{F_\infty}}{\sqrt{3F_\infty}} - \operatorname{arctg} \frac{1}{\sqrt{3}}\right\}. \qquad (6.20)$$

Assuming that $F' = 1$ at the point where $u = u_{max}$ it is easy to find that $F_\infty = 1.7818$.

Equations (6.18) and (6.20) may be regarded as parametric equations for the determination of the function $F = F(\varphi)$. With a series of pregiven arbitrary values of the function F, we obtain from these equations a series of mutually corresponding values of F' and φ with the help of which we can draw the curve representing the nondimensional velocity $F = u/u_m$ as a function of the variable φ. This curve is shown in Fig. 6.1.

For a total solution of the dynamic problem, we determine the values of the constants A and B from Eqs. (6.13) and (6.15):

$$A = \sqrt{\frac{K}{4\rho^2 \nu J_1}}, \quad B = \frac{1}{2}\sqrt[4]{\frac{K}{4\rho^2 \nu^3 J_1}}. \qquad (6.21)$$

The velocity components in a plane semilimited jet are thus determined by the expressions

$$
\left.\begin{aligned}
u &= \frac{1}{2}\sqrt{\frac{K}{\rho^2\nu J_1}}\,x^{-1/2}F'(\varphi),\\
v &= \frac{1}{4}\sqrt[4]{\frac{4\nu K}{\rho^2 J_1}}\,x^{-3/4}(3\varphi F' - F)\\
&\left(\varphi = \frac{y}{2}\sqrt{\frac{K}{4\rho^2\nu^3 J_1 x^3}}\right).
\end{aligned}\right\} \qquad (6.22)
$$

Let us finally give the laws of variation along the jet of the mass flow rate G per second, the momentum J_x, the kinetic energy flux E

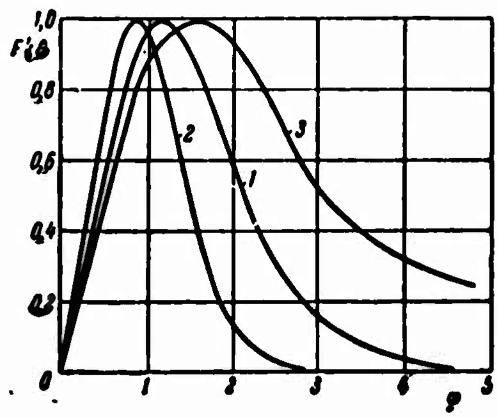

Fig. 6.1 Universal velocity and temperature distributions in a semilimited jet.
1. $u/u_m = \Delta T/\Delta T_m$ ($Pr = 1$); 2. $\Delta T/\Delta T_m$ ($Pr = 2$); 3. $\Delta T/\Delta T_m$ ($Pr = 0.5$).

and the frictional stress τ_w at the plate:

$$
G = \int_0^\infty \rho u\, dy = F_\infty \sqrt[4]{\frac{4K\rho^2\nu x}{J_1}} \sim x^{1/4}, \qquad (6.23)
$$

$$
J_x = \int_0^\infty \rho u^2\, dy = J_2 \sqrt[4]{\frac{K^3}{4\rho^2\nu J_1^3 x}} \sim x^{-1/4} \quad \left(J_2 = \int_0^\infty F'^2 d\varphi\right), \qquad (6.24)
$$

$$
E = \frac{1}{2}\int_0^\infty \rho u^3 dy = \frac{K}{4\rho J_1}\sqrt[4]{\frac{K}{4\rho^2\nu J_1 x^3}} \sim x^{-3/4}, \qquad (6.25)
$$

$$
\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \frac{1}{4}F''_w \sqrt[4]{\frac{K^3}{\rho^2\nu J_1^3 x^5}} \sim x^{-5/4}. \qquad (6.26)
$$

It follows from Eqs. (6.23) and (6.24) that for a plane semilimited jet expanding along a plate the product of momentum and flow

rate per second remains constant:

$$J_x G = F_\infty \frac{J_2}{J_1} K = \text{const.}$$

Let us now consider the solution of the thermal problem. The equation of heat propagation

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = a \frac{\partial^2 T}{\partial y^2} \tag{6.27}$$

will be integrated with three different types of boundary conditions [53]:

(a) the wall temperature T_w and the temperature T_∞ of the fluid at rest are equal:

$$\left.\begin{array}{ll} \Delta T = 0 & \text{if } y = 0, \\ \Delta T = 0 & \text{if } y = \infty \\ (\Delta T = T - T_\infty = T - T_w); & \end{array}\right\} \tag{6.28a}$$

(b) a nonheatconducting wall:

$$\left.\begin{array}{ll} \frac{\partial \Delta T}{\partial y} = 0 & \text{if } y = 0, \\ \Delta T = 0 & \text{if } y = \infty \\ (\Delta T = T - T_\infty); & \end{array}\right\} \tag{6.28b}$$

(c) a constant wall temperature which differs from the temperature of the fluid far away from the plate:

$$\left.\begin{array}{ll} \Delta T = \Delta T_w & \text{if } y = 0, \\ \Delta T = 0 & \text{if } y = \infty \\ (\Delta T = T - T_\infty;\ \Delta T_w = T_w - T_\infty). & \end{array}\right\} \tag{6.28c}$$

Let us consider the solution of the energy equation with the above types of boundary conditions, one after the other.

(a) In the case of similar boundary conditions for velocity and temperature difference [Conditions (6.28)] a unique solution of the thermal problem requires (as in the case of the free jet) the application of an integral condition of conservation of some quantity. This is firstly necessary in order to obtain a nontrivial solution (since the boundary conditions for the temperature difference contain the possibility of zero temperature at the body surface and away from it). Secondly, the integral invariant helps us in determining the

value of the constant of self-similarity γ in the temperature equation.

A conclusion analogous to the above for the condition K = const will in the general case result in the equation

$$\frac{d}{dx}\int_0^\infty u\,\Delta T\left(\int_0^y u\,dy\right)dy = a\int_0^\infty u\,\frac{\partial\,\Delta T}{\partial y}\,dy,$$

which does not enable us to draw any general conclusion with respect to the invariant or the problem of the value of the constant γ. Let us therefore first consider the simple particular case of a Prandtl number equal to unity. Just as in other cases (with symmetrical boundary conditions for velocity and temperature) the velocity and temperature distributions are similar if Pr = 1. In this case, the integral in the righthand side of the above equation vanishes. Analogously to the dynamic problem, the thermal problem, with Pr - 1, also possesses a general (independent of the supposition of self-similarity) invariant of the form

$$K_T^1 = \int_0^\infty \rho c_p u\,(T - T_\infty)\left(\int_0^y \rho u\,dy\right)dy = \text{const.} \qquad (6.29)$$

Hence it follows in particular that the constants α and γ are equal to one another, i.e., $\alpha = \gamma = -\frac{1}{2}$.

Here

$$\Delta T_m = \frac{K_T^1}{c_p K}\,u_m = \frac{K_T^1}{c_p K}\,Ax^a$$

and finally

$$\Delta T = T - T_\infty = \Gamma x^{-1/2} F'(\varphi) = \frac{K_T^1}{2\rho c_p \sqrt{\nu K I_1 x}}\,F'(\varphi)$$

and

$$\theta(\varphi) = \frac{T - T_\infty}{T_m - T_\infty} = F'(\varphi). \qquad (6.30)$$

With other Prandtl numbers different from unity, the determination of the temperature distribution requires a preliminary choice of

the value of the constant γ. In analogy to the great number of problems on self-similar flows (summarized below in Table 7.1), we assume that in the problem considered and values of the constant exponents α and γ of the expressions $u_m = Ax^\alpha$ and $\Delta T_m = \Gamma x^\gamma$, are equal. In the given case, just as with Pr = 1, we must therefore assume that $\alpha = \gamma = -1/2$. The problem can then be reduced to a simple numerical integration of a differential equation for the function $\theta(\varphi)$, which has the form

$$\frac{1}{Pr}\theta'' + F\theta' + 2F'\theta = 0 \qquad (6.31)$$

with the boundary conditions $\theta(0) = \theta(\infty) = 0$.

The validity of the above supposition on the equality of the values of α and γ outside the dependence on the Prandtl number is proven by the fact that with this value of γ a numerical calculation permits the construction of an integral curve $\theta(\varphi)$ for each value of Pr, which satisfied the boundary conditions and the additional condition that the maximum value θ_{max} is equal to unity with a certain value of the argument φ_m. For the calculation one must choose a value of $\theta'(0)$ which satisfies the condition $\theta_{max} = 1$. The results of a calculation for two values of Pr are shown in Fig. 6.1. The same numerical calculation for Pr = 1 yields a practical coincidence with the exact solutions.

To illustrate the influence of the Prandtl number on the temperature distribution, the following table gives the approximate values of φ_m and $\theta'(0)$ for several values of Pr:

Pr	0,5	0,75	1,0	2,0
φ_m	1,50	1,26	1,10	0,80
$\theta'(0)$	1,00	1,07	1,13	1,50

The data given in the table and in Fig. 6.1 are, as in other analogous cases, indicating the increase in thickness of the boundary layer of both the jet flow and the flow near the wall with a decrease of the Prandtl number. As regards the integral invariant satisfying a nontrivial solution (in the calculation this is achieved through a choice of $\theta'(0) \neq 0$) and permitting the determination of the still unknown value of the constant $\Gamma\,(\mathrm{Pr})$ for the self-similar thermal (and dynamic) problem of the plane semilimited jet the following equations hold true:

$$\int_0^\infty u^{3/2}\,dy = \text{const}, \quad \int_0^\infty \Delta T^{3/2}\,dy = \text{const}.$$

The second of these integrals must be taken to the given parameters of the problem and thus enable us to find $\Gamma\,(\mathrm{Pr})$.

We denote the integral as follows:

$$\int_0^\infty \Delta T^{3/2}\,dy = K_T = \text{const}.$$

In this case the value of the constant Γ will be equal to

$$\Gamma\,(\mathrm{Pr}) = \left[\frac{BK_T}{\int_0^\infty \theta^{3/2}\,d\varphi}\right]^{1/\beta}$$

and, with $\beta = -\frac{3}{4}$, $\gamma = -\frac{1}{2}$ we finally obtain

$$\Gamma\,(\mathrm{Pr}) = \left[\int_0^\infty \theta^{3/2}\,d\varphi\,(BK_T)^{-1}\right]^{4/3}.$$

In order to finish the problem completely we also determine the law of variation of the total flux of excessive heat content along the jet:

$$Q = \int_0^\infty \rho c_p u\,\Delta T\,dy = \rho c_p \frac{A\Gamma}{B} x^{\alpha+\gamma-\beta} \int_0^\infty F'\theta\,d\varphi \sim x^{-1/4}. \qquad (6.32)$$

We see from this formula that the function $Q \sim x^{-1/4}$ maintains its form which is equal to that of the momentum flux $J_x \sim x^{-1/4}$, independently of

the Prandtl number.

b) In the case of the expansion of a semilimited jet along an nonheatconducting plate the integral invariant of the thermal problem is obvious from the physical point of view. It is easily obtained also from Eq. (6.27) by integrating it across the jet. In this way, the thermal integral invariant in the form

$$Q = \int_0^\infty \rho c_p u \, \Delta T \, dy = \text{const} \qquad (6.33)$$

corresponds to the condition of conservation of the total flux of excessive heat content in the jet.

Suggesting

$$\frac{\Delta T}{\Delta T_m} = \theta(\varphi), \quad T_m - T_\infty = \Delta T_m = \Gamma x^\gamma, \qquad (6.34)$$

for the solution of the problem, we transform the equation of heat propagation (6.27) to an ordinary differential equation for the sought function $\theta(\varphi)$:

$$\theta'' + \Pr(F\theta' - 4\gamma F'\theta) = 0, \qquad (6.35)$$

which must be integrated under the boundary conditions

$$\left.\begin{aligned} \theta' = 0 \text{ if } \varphi = 0, \\ \theta = 0 \text{ if } \varphi = \infty. \end{aligned}\right\} \qquad (6.36)$$

With the help of the integral condition (6.33) we can determine the values of the constants γ and Γ

$$\gamma = \beta - \alpha = -\frac{1}{4}, \quad \Gamma = \frac{BQ}{\rho c_p A J_2} = \frac{Q}{c_p J_2} \sqrt[4]{\frac{J_1}{4\rho^3 \nu K}} \left[J_2(\Pr) = \int_0^\infty F'\theta \, d\varphi\right].$$

With the value of $\gamma = -\frac{1}{4}$ obtained we can rewrite Eq. (6.35) in the form

$$\theta'' + \Pr(F\theta)' = 0. \qquad (6.37)$$

When we take the boundary conditions (6.36) into account, it is easy to solve the latter equation

$$\theta(\varphi) = \exp\left(-\Pr \int_0^\varphi F \, d\varphi\right). \qquad (6.38)$$

Thus we finally have for the surplus temperature

$$\Delta T = T - T_\infty = \frac{Q}{c_p J_3} \sqrt[4]{\frac{J_1}{4\rho^2 \nu K}}\, x^{-1/4} \exp\left(-\mathrm{Pr}\int_0^\varphi F\, d\varphi\right). \qquad (6.39)$$

(In this variant of the boundary conditions, the sought value of the wall temperature is equal to the factor of the power function in the last formula). The temperature distribution calculated from Eq. (6.39) is represented in Fig. 6.2.

c) Finally, in order to solve the thermal problem of a jet which expands along a wall with constant temperature (different from the temperature of the fluid at $y = \infty$) we put (for $\gamma = 0$)

$$\frac{\Delta T}{\Delta T_w} = \frac{T - T_\infty}{T_w - T_\infty} = \theta(\varphi); \qquad (6.40)$$

we then obtain instead of (6.27) and (6.28c) the equation

$$\theta'' + \mathrm{Pr}\cdot F \cdot \theta' = 0 \qquad (6.41)$$

with the boundary conditions:

$$\theta = 1 \text{ if } \varphi = 0, \quad \theta = 0 \text{ if } \varphi = \infty. \qquad (6.42)$$

Integrating Eq. (6.41) twice, with the boundary conditions (6.42) taken into account, we arrive at a solution in the form

$$\theta(\varphi) = 1 - \left[\int_0^\varphi \exp\left(-\mathrm{Pr}\int_0^\varphi F\, d\varphi\right) d\varphi\right]\left[\int_0^\infty \exp\left(-\mathrm{Pr}\int_0^\varphi F\, d\varphi\right)^{-1} d\varphi\right]. \qquad (6.43)$$

The distribution of the excessive nondimensional temperature in a jet cross section is shown in Fig. 6.3.

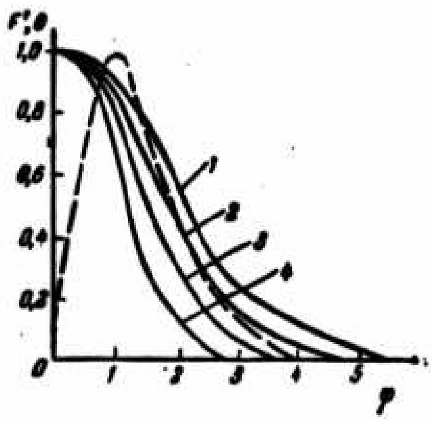

Fig. 6.2 Temperature distribution in a jet moving along a non-heatconducting wall (- - - - velocity). 1. Pr = 0,5; 2. Pr = 0,75; 3. Pr = 1,0; 4. Pr = 2,0.

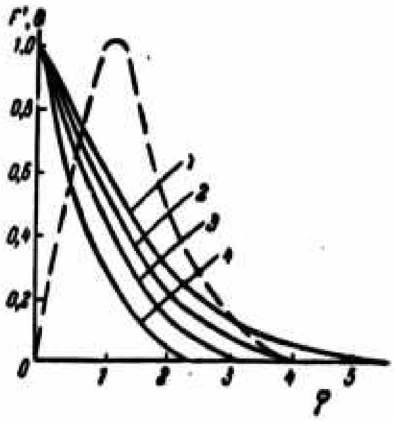

Fig. 6.3. Temperature distribution in a jet moving along a wall of constant temperature (- - - - velocity). 1. Pr = 0,5; 2. Pr = 0,75; 3. Pr = 1,0; 4. Pr = 2,0.

At last we calculate the flux of excessive heat content in the jet and the heat transfer through the plate:

$$Q = \int_0^\infty \rho c_p u \,\Delta T \, dy = 2c_p (T_w - T_\infty) \sqrt[4]{\frac{\rho^2 \nu K x}{\lambda J_1}} \int_0^\infty F'\theta \, d\varphi \sim x^{1/4}. \qquad (6.44)$$

The quantity Q varies with the length in the same way as the fluid flow rate in the jet. The mean value of the Nusselt number, averaged over the length of the plate, is given by

$$\overline{\mathrm{Nu}} = \frac{\bar{\alpha} x}{\lambda} = \frac{1}{\lambda} \int_0^x \alpha(x)\, dx, \text{ where } \alpha(x) = \frac{\lambda}{T_w - T_\infty} \frac{\partial \Delta T}{\partial y}\bigg|_{y=0}.$$

Using the solution of (6.43) obtained, we find

$$\overline{\mathrm{Nu}} = \frac{1}{8} \sqrt[4]{\frac{K x}{4\rho^2 \nu^3 J_1}} \left[\int_0^\infty \exp\left(-\mathrm{Pr} \int_0^\varphi F \, d\varphi \right) d\varphi \right]^{-1} \sim x^{3/4}. \qquad (6.45)$$

The mean value of the heat transfer coefficient, $\tilde{\alpha} \sim x^{-1/4}$ will, as usual, drop as the Nusselt number increases along the plate.

6.2 JET ALONG A STRAIGHT ROUND CONE

Let us now consider the problem of a slightly twisted laminar jet of an incompressible fluid, discharged from a small nozzle and expanding along the surface of a straight round cone with a vertex angle of

2ω (see schematic diagram of Fig. 6.4).

We choose an orthogonal system of coordinates in which the Ox axis is directed along the generatrix of the cone and the Oy axis is normal to it, while the coordinate θ measures the angle around the axis of the cone. The origin of coordinates is allowed to coincide with the vertex of the cone.

In the coordinate system chosen the Navier-Stokes equations and the energy equation for an axisymmetric motion, in the absence of volume forces and heat sources, have the following forms [54]:

$$\left.\begin{aligned}
&u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-\frac{w^2\sin\omega}{x\sin\omega+y\cos\omega}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\\
&\quad+\nu\left\{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{1}{x\sin\omega+y\cos\omega}\times\right.\\
&\quad\left.\times\left[\left(2\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\sin\omega+\frac{\partial u}{\partial y}\cos\omega\right]\right\},\\
&u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}-\frac{w^2\cos\omega}{x\sin\omega+y\cos\omega}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\\
&\quad+\nu\left\{\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{1}{x\sin\omega+y\cos\omega}\times\right.\\
&\quad\left.\times\left[\left(\frac{\partial u}{\partial x}+2\frac{\partial v}{\partial y}\right)\cos\omega+\frac{\partial v}{\partial x}\sin\omega\right]\right\},\\
&u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+\frac{u\sin\omega+v\cos\omega}{x\sin\omega+y\cos\omega}w=\\
&\quad=\nu\left\{\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{1}{x\sin\omega+y\cos\omega}\times\right.\\
&\quad\left.\times\left[\frac{\partial w}{\partial x}\sin\omega+\frac{\partial w}{\partial y}\cos\omega-\frac{w}{x\sin\omega+y\cos\omega}\right]\right\},\\
&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{u\sin\omega+v\cos\omega}{x\sin\omega+y\cos\omega}=0;\\
&u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=a\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\right.\\
&\quad\left.+\frac{1}{x\sin\omega+y\cos\omega}\left(\frac{\partial T}{\partial x}\sin\omega+\frac{\partial T}{\partial y}\cos\omega\right)\right].
\end{aligned}\right\}\tag{6.46}$$

An estimation of the terms of Eqs. (6.46) usually carried out in the boundary layer theory (or a transition to the boundary layer equations in some other way) yields

$$\left.\begin{aligned}
&u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-\frac{w^2}{x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2},\\
&\frac{\rho w^2}{x}\operatorname{ctg}\omega=\frac{\partial p}{\partial y},\\
&u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+\frac{uw}{x}=\nu\frac{\partial^2 w}{\partial y^2},\\
&\frac{\partial}{\partial x}(xu)+\frac{\partial}{\partial y}(xv)=0,
\end{aligned}\right\}\tag{6.47}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=a\frac{\partial^2 T}{\partial y^2}.\tag{6.48}$$

If we restrict ourselves to the case of a slightly twisted jet, on the basis of considerations, analogous to those of §4.4 when considering the problem of a free fan-type twisted source jet, we must omit the terms $-w^2/x$ and $-\frac{1}{\rho}\frac{\partial p}{\partial x}$. in the first equation of System (6.47).

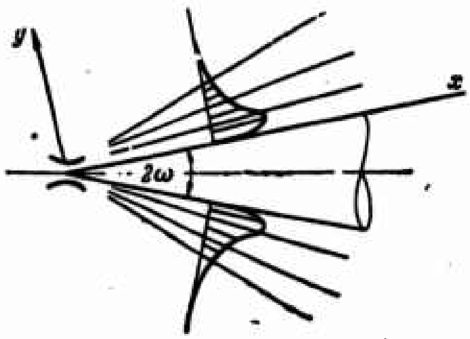

Fig. 6.4. Schematic representation of the expansion of a jet along a cone.

The system of the boundary layer equations (6.47) for the dynamic problem must be integrated with the boundary conditions

$$\left.\begin{aligned}
u=v=w=0 \text{ if } \quad y=0,\\
u=w=0, \text{ if } \quad y=\infty.
\end{aligned}\right\}\tag{6.49}$$

Since also here we are concerned with a source jet, the boundary conditions (6.49) must be completed by integral conditions of conservation of the form

$$\int_0^\infty \rho x u^2\left(\int_0^y \rho x u\, dy\right)dy=K=\text{const},\tag{6.50}$$

$$\int_0^\infty \rho x^2 uw \left(\int_0^y \rho x u \, dy\right) dy = N = \text{const}, \tag{6.51}$$

which are derived analogously as those in the previous section (see also [202]). Assuming $K \neq 0$ and $N \neq 0$, we find a nontrivial solution.

We put as usual for the solution of the dynamic problem

$$\frac{u}{u_m} = F'(\varphi), \quad \frac{w}{w_m} = \Phi(\varphi), \quad \frac{p_\infty - p}{p_\infty - p_m} = P(\varphi), \tag{6.52}$$

where

$$u_m = Ax^\alpha, \quad w_m = Cx^\varepsilon, \quad p_\infty - p_m = Dx^\delta, \quad \varphi = Byx^\beta.$$

When we transform the equation of the dynamic problem (6.47) with the help of Eqs. (6.52) we arrive at a system of ordinary differential equations:

$$\left.\begin{aligned} &F''' + FF'' + 2F'^2 = 0, \\ &P' = -\Phi^2, \\ &\Phi'' + F\Phi' + 2F'\Phi = 0 \end{aligned}\right\} \tag{6.53}$$

with the boundary conditions

$$\left.\begin{aligned} &F = F' = \Phi = 0 \quad \text{if } \varphi = 0, \\ &F' = \Phi = P = 0 \quad \text{if } \varphi = \infty. \end{aligned}\right\} \tag{6.54}$$

The values of the constants of self-similarity and the constants A, B, C, D, obtained in this transformation of equations and application of the integral invariants (6.50) and (6.51) are equal to

$$\alpha = -\frac{3}{2}, \quad \beta = -\frac{5}{4}, \quad \varepsilon = -\frac{5}{2}, \quad \delta = -\frac{19}{4}, \tag{6.55}$$

and also

$$\left.\begin{aligned} &A = \sqrt{\frac{3K}{4\rho^2\nu J_1}}, \quad B = \sqrt[4]{\frac{27K}{64\rho^2\nu^3 J_1}}, \quad C = \frac{N}{\rho}\sqrt{\frac{3}{4\nu K J_1}}, \\ &D = \left(\frac{N}{K}\right)^2 \operatorname{ctg}\omega \sqrt[4]{\frac{3K^3}{4\rho^2\nu J_1^3}} \quad \left(J_1 = \int_0^\infty FF'^2 \, d\varphi\right). \end{aligned}\right\} \tag{6.56}$$

Comparing the first and the third equations of System (6.53) and the boundary conditions for the functions F' and Φ it follows that

$$\Phi = F'.$$

Moreover, the problem of finding the function $F'(\varphi)$ is analogous

to the problem considered in detail in the preceding section (on the plane semilimited jet). The solution can therefore be written down immediately

$$F' = \Phi = \frac{2}{3}\left(F_\infty^{3/2} F^{1/2} - F^2\right), \tag{6.57}$$

where

$$\varphi = \frac{1}{2F_\infty}\left\{\ln \frac{F + \sqrt{FF_\infty} + F_\infty}{(\sqrt{F} - \sqrt{F_\infty})^2} + \right.$$
$$\left. + 2\sqrt{3}\left(\operatorname{arctg}\frac{2\sqrt{F} + \sqrt{F_\infty}}{\sqrt{3F_\infty}} - \operatorname{arctg}\frac{1}{\sqrt{3}}\right)\right\} \tag{6.58}$$

(as also previously $F_\infty = 1{,}7818$)

Thus, we finally have for the velocity components and the pressure

$$\left.\begin{aligned} u &= \frac{A}{x^{6/4}} F'(\varphi), \\ v &= \frac{A}{4Bx^{5/4}}(5\varphi F' - 3F), \\ w &= \frac{C}{x^{6/4}} F'(\varphi), \\ p_\infty - p &= \frac{D}{x^{12/4}} \int_\varphi^\infty F'^2\, d\varphi. \end{aligned}\right\} \tag{6.59}$$

We use the expressions obtained for the velocity components in order to calculate the mass flow rate G, the momentum J_x of the jet, the moment of momentum M and the total flux of kinetic energy E:

$$G = \int_0 \rho u\, ds = 2\pi\rho \sin\omega \frac{A}{B} F_\infty x^{3/4} \sim x^{3/4},$$

$$J_x = \int_0 \rho u^2\, ds = 2\pi\rho \sin\omega \frac{A^2}{B} x^{-3/4} \int_0^\infty F'^2\, d\varphi \sim x^{-3/4},$$

$$M = \int_0 \rho x u w \sin\omega\, ds = 2\pi\rho \sin^2\omega \frac{AC}{B} x^{-3/4} \int_0^\infty F'^2\, d\varphi \sim x^{-3/4},$$

$$E = \frac{1}{2}\int_0 \rho u^3\, ds = \pi\rho \sin\omega \frac{A^3}{B} x^{-9/4} \int_0^\infty F'^3\, d\varphi \sim x^{-9/4}.$$

Just as in paper [202] the product of flow rate and total momentum GJ_x or the product of flow rate and moment of momentum GM and also a product of the form G^3E remain constant along the jet in conic sec-

tions orthogonal to the surface of the cone.

In the coordinate system chosen the expressions for the velocity components coincide with those obtaindd in the paper by M.S. Tsukker [202] for the case of $\omega = \frac{\pi}{2}$. This limiting case corresponds to the efflux of a fan-type, twisted semilimited jet. The pressure gradient vanishes as $\omega \to \frac{\pi}{2}$ The presence of a pressure gradient $(\frac{\partial p}{\partial y} \neq 0)$ is a characteristic peculiarity of the problem considered in the case of $\omega < \frac{\pi}{2}$.

Let us now turn to the solution of the thermal problem. Considering the three types of boundary conditions (6.28a-c) we write the solutions of the energy equation (6.48) in the following form

$$\text{a) } \Delta T = \Gamma x^{\gamma}\theta_a(\varphi), \tag{6.60a}$$
$$\text{b) } \Delta T = \Gamma x^{\gamma}\theta_6(\varphi), \tag{6.60b}$$
$$\text{c) } \Delta T = \Delta T_w\theta_s(\varphi). \tag{6.60c}$$

In order to obtain self-similar solutions of the energy equations (6.48) with the boundary conditions (6.28) we complete them by the integral invariants and the values of the constant of self-similarity, γ. We do this subsequently for all three forms of the temperature boundary conditions.

a) Under symmetrical boundary conditions for the velocity and temperature we assume, as in the problem of the plane semilimited jet, that

$$\gamma = \alpha = -\frac{3}{2}, \quad K_T = \int_0^\infty (\Delta T)^{4/3} x\,dy = \text{const.}$$

In the particular case of a Prandtl number equal to unity, we also give as above an expression for a more general invariant:

$$K_T^1 = \int_0^\infty \rho c_p u\,\Delta T \left(\int_0^y \rho u x\,dy\right) x\,dy = \text{const.}$$

b) For a nonheatconducting wall

$$Q_6 = 2\pi \sin\omega \int_0^\infty \rho c_p u\,\Delta T x\,dy = \text{const}, \quad \gamma = -\frac{3}{4}.$$

c) For a wall of constant temperature $\gamma = 0$.

After the transition to the self-similar equations, we arrive again at the same ordinary differential equations as in the case of the plane semilimited jet:

$$\theta_a'' + \Pr\,(F\theta_a' + 2F'\theta_a) = 0,$$
$$\theta_б'' + \Pr\,(F\theta_б)' = 0,$$
$$\theta_в'' + \Pr\,F\theta_в' = 0,$$

with the same boundary conditions. The solutions for the fan-type semilimited jet will therefore have the same form as those obtained for the plane jet. Only the values of the constants will be new, which is due to the other form of the continuity equation.

The final results read as follows:

a)
$$\Delta T = \left[\frac{BK_T}{\int\limits_0^\infty \theta_a^{3/2}\,d\varphi}\right]^{\frac{2}{3}} x^{-3/4}\theta_a(\varphi) \tag{6.61}$$

$$\left(\Delta T = \frac{K_T^1}{2\rho c_p}\sqrt{\frac{3}{\nu K J_1}}\; x^{-3/4}F'(\theta) \text{ for } \Pr = 1\right),$$

b)
$$\Delta T = \frac{Q_б}{2\pi\rho c_p J_2 \sin\omega}\sqrt[4]{\frac{3J_1}{4\rho^2\nu K}}\, x^{-3/4}\exp\left(-\Pr\int\limits_0^\varphi F\,d\varphi\right), \tag{6.62}$$

c)
$$\Delta T = \Delta T_w\left\{1 - \left[\int\limits_0^\varphi \exp\left(-\Pr\int\limits_0^\varphi F\,d\varphi\right)d\varphi\right]\times\right.$$
$$\left.\times\left[\int\limits_0^\infty \exp\left(-\Pr\int\limits_0^\varphi F\,d\varphi\right)d\varphi\right]^{-1}\right\}. \tag{6.63}$$

It is easy to calculate the variation of the flux of excessive heat content under the conditions of the problems a) and c):

$$Q_a = 2\pi\rho c_p \sin\omega\,\frac{A\Gamma}{B}\,x^{-1/4}\int\limits_0^\infty F'\theta\,d\varphi \sim x^{-1/4}$$

in the case of a wall temperature which equals that of the fluid away from it (with an arbitrary value of Pr) and

$$Q_в = 2\pi\rho c_p \sin\omega\,(T_w - T_\infty)\sqrt[4]{\frac{4\nu K}{3\rho^2 J_1}}\,x^{3/4}\int\limits_0^\infty F'\theta_в\,d\varphi \sim x^{3/4}$$

with a constant wall temperature ($T_w \neq T_\infty$). The latter relationship is

again the same as in the case of the efflux of a jet of fluid.

Finally, for a mean cone (with respect to the lateral surface) the value of the Nusselt number in case c) is obtained as follows:

$$\overline{\mathrm{Nu}} = -4B\theta_s'(0)\,x^{-1/4}.$$

In the limiting case of a vertex angle of $\omega = \frac{\pi}{2}$ our problem on the expansion of a jet along the surface of a cone becomes the problem of the fan-type semilimited jet.

In an analogous way B.P. Ustimenko, in his paper [198] solved the problem of the expansion of a laminar semilimited jet along a cylinder.

6.3 A JET ALONG A POROUS WALL

Another problem related to those considered in the preceding sections is the problem of the expansion of a plane-parallel laminar jet of incompressible fluid discharged from a thin slit oriented along the axis Oz and moving along a porous plate in the direction of the axis Ox. Fluid of the boundary layer may flow off through the plate or be injected through it into this layer. The physical properties of the fluid supplied (or withdrawn) through the plate are assumed to be the same as in the jet. Outside the jet, just as in the usual semilimited jet, we assume a fluid of the same properties which (at infinity) is at rest.

The solution is based on the system of equations (6.1) of the laminar plane boundary layer

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{6.1}$$

with the following boundary conditions

$$\left.\begin{array}{lll} u = 0, & v = v_w = v_0 x^{\varkappa} & \text{if } y = 0, \\ u = 0 & & \text{if } y = \infty. \end{array}\right\} \tag{6.64}$$

In this way, as usually, the longitudinal velocity component vanishes at the surface of the porous plate; the transverse velocity

component is assumed to be a power function of the coordinate x.

In order to obtain a nontrivial solution we use an integral condition whose derivation is analogous to that of the invariant K in the problem of the semilimited jet. For the jet along the porous plate, this condition has the form

$$\int_0^\infty u^2\left(\int_0^y u\,dy\right)dy-\int_0^x v_w\left(\int_0^\infty u^2\,dy\right)dx=L=\mathrm{const}. \qquad (6.65)$$

The components of the velocity vector can be written in the form of power series of the nondimensional parameter q:

$$q=\frac{v_w v^2}{L}=q_0x^{\varkappa}, \qquad (6.66)$$

which is assumed sufficiently small:

$$\left.\begin{aligned} u&=Ax^{\alpha}F_0'(\varphi)+qA_1x^{\alpha_1}F_1'(\varphi)+\ldots,\\ v&=v_w-\frac{A}{B}x^{\alpha-\beta-1}[(\alpha-\beta)F_0+\beta\varphi F_0']-\\ &\quad-q\frac{A_1}{B}x^{\alpha_1-\beta-1}[(\alpha_1-\beta)F_1+\beta\varphi F_1']-\ldots \end{aligned}\right\} \qquad (6.67)$$

Similarly as in other cases we have also here $\varphi=Byx^{\beta}$; the prime denotes the differential quotient with respect to the variable φ.

We now substitute Eq. (6.67) in the initial equations. The coefficients of the same powers of the parameter q are then set equal to zero. As a result, we obtain a system of ordinary differential equations for the determination of the functions F_0, F_1, ...:

$$F_0'''+\frac{A}{vB^2}\left(\frac{\alpha+1}{2}F_0F_0''-\alpha F_0'^2\right)=0, \qquad (6.68)$$

$$F_1'''+\frac{A}{vB^2}\left[\frac{\alpha+1}{2}F_0F_1''-\frac{3\alpha+2\varkappa+1}{2}F_0'F_1'+(\varkappa+1)F_0''F_1\right]-$$
$$-\frac{LA}{v^3A_1B}F_0''=0, \qquad (6.69)$$

Instead of the integral condition (6.65) we obtain

$$A^3x^{3\alpha}\int_0^\infty F_0'^2\left(\int_0^y F_0'\,dy\right)dy=L \qquad (6.70)$$

and

$$A_1 x^{2\alpha+\alpha_1}\int\limits_0^\infty F_0'^2\left(\int\limits_0^y F_1'\,dy\right)dy + 2A_1 x^{2\alpha+\alpha_1}\int\limits_0^\infty F_0'F_1'\left(\int\limits_0^y F_0'\,dy\right)dy - \qquad (6.71)$$
$$- \frac{L}{\nu^3}\int\limits_0^x x^{2\alpha}\left(\int\limits_0^\infty F_0'^2\,dy\right)dy = 0,$$

In the derivation of the equations and the integral conditions, we assumed that

$$\beta = \frac{\alpha-1}{2}, \quad \alpha_1 = \frac{\alpha+1}{2}. \qquad (6.72)$$

From Eq. (6.70) we have in addition to this, $3\alpha = 2\beta$. So we obtain

$$\alpha = -\frac{1}{2}, \quad \beta = -\frac{3}{4}, \quad \alpha_1 = \frac{1}{4}. \qquad (6.73)$$

Let us also put

$$A = \frac{1}{2}\left(\frac{L}{\nu}\right)^{1/4}, \quad B = \frac{1}{2}\left(\frac{L}{4\nu^3}\right)^{1/4}, \quad A_1 = \frac{1}{2}\,\frac{L}{\nu^3}\left(\frac{L}{4\nu^3}\right)^{1/4}, \qquad (6.74)$$

such that the equations, boundary and integral conditions for the zero<u>th</u> approximation take the form

$$F_0''' + F_0F_0'' + 2F_0'^2 = 0, \qquad (6.75)$$

$$\left.\begin{array}{lll} F_0 = F_0' = 0 & \text{if} & \varphi = 0, \\ F_0' = 0 & \text{if} & \varphi = \infty, \end{array}\right\} \qquad (6.76)$$

$$\int\limits_0^\infty F_0F_0'^2\,d\varphi = 1. \qquad (6.77)$$

Analogously, we have for the first approximation, to which we restrict ourselves

$$F_1''' + F_0F_1'' + (1-4\varkappa)F_0'F_1' + 4(1+\varkappa)F_0''F_1 = 4F_0'', \qquad (6.78)$$

$$\left.\begin{array}{lll} F_1 = F_1' = 0 & \text{if} & \varphi = 0, \\ F_1' = 0 & \text{if} & \varphi = \infty, \end{array}\right\} \qquad (6.79)$$

$$\int\limits_0^\infty F_0'\left[F_0'\left(F_1 - \frac{4}{3}\right) + 2F_0F_1'\right]d\varphi = 0. \qquad (6.80)$$

The solution of the equations in zero<u>th</u> approximation, with the corresponding boundary conditions, are well known (cf.Section (6.1):

$$\left.\begin{array}{l} F_0' = \frac{2}{3}F_{0\infty}^2 z(1-z^3) \quad \left(z^3 = \frac{F_0}{F_{0\infty}}\right), \\ \varphi = \frac{1}{2F_{0\infty}}\ln\frac{1+z+z^2}{(1-z)^2} + \frac{\sqrt{3}}{F_{0\infty}}\operatorname{arctg}\frac{\sqrt{3}\cdot z}{2+z}. \end{array}\right\} \qquad (6.81)$$

Consequently, the coefficients in the equations of the first

approximation (6.78) are determined. What remains arbitrary is the value of the constant κ (the exponent in the law governing the sinks and sources).

We consider $\varkappa = 0$. as the simplest particular case. This corresponds to sources or sinks of constant intensity along the plate. In this case Eq. (6.78) takes the form

$$F_1''' + F_0F_1'' + F_0'F_1' + 4F_0''F_1 = 4F_0''. \qquad (6.82)$$

We introduce the new independent variable $z\left(z^2 = \frac{F_0}{F_{0\infty}}\right)$ instead of φ. Analogously as in paper [19], we then obtain the equation (prime: differentiation with respect to z)

$$(1-z^2)^2F_1'''(z) - 6z^2(1-z^2)F_1''(z) + 24(1-4z^2)F_1(z) = 24(1-4z^2) \qquad (6.83)$$

with the boundary conditions:

$$\left.\begin{aligned} F_1 = F_1' = 0 \quad &\text{if} \quad z = 0, \\ \lim\,(1-z^2)F_1'(z) = 0 \quad &\text{if} \quad z \to 1. \end{aligned}\right\} \qquad (6.84)$$

Equation (6.83) has a singular point at $z = 1$. We shall therefore solve it in the following way. Since in a considerable range of variations of z, e.g., $0 < z < 0{,}9$, there are no singularities, the solution of Eq. (6.83), which can be represented in the form

$$[(1-z^2)^2F_1'']' = 24(1-4z^2)(1-F_1), \qquad (6.83a)$$

can be found by the method of successive approximations (as in paper [19]). Near the point $z = 1$ the solution of this equation is sought in the form of a power series of $1 - z$. With the second boundary condition of (6.84) the general solution for the range $0{,}9 < z < 1$ has the form

$$\begin{aligned} F_1(z) = 1 + C_1\left[1 - z - 2(1-z)^2 + \frac{4}{3}(1-z)^3 - \frac{1}{3}(1-z)^4 + \ldots\right] + \\ + C_2\left\{\left[1 - z + 2(1-z)^2 + \frac{4}{3}(1-z)^3 - \frac{1}{3}(1-z)^4\right]\ln(1-z) + \right. \\ \left. + \left[-\frac{1}{8} + 4(1-z)^2 - 3{,}166(1-z)^3 + 1{,}638(1-z)^4 + \ldots\right]\right\} + \cdots \end{aligned} \qquad (6.85)$$

The constants C_1 and C_2 are determined when Solution (6.85) is "sewed together" with the solution for the range $0 \leqslant z \leqslant 0{,}9$. obtained by the method of successive approximations.

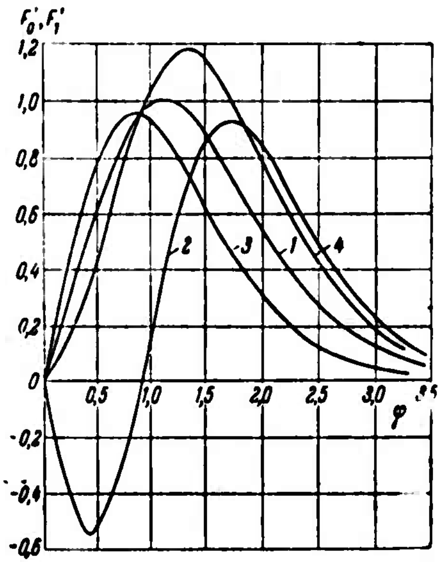

Fig. 6.5. Universal velocity distributions for a semilimited jet along a porous wall. 1) (u/u_{m0}) for an impermeable wall; 2) the function $F_1'(\varphi)$; 3) $(u/u_{m0})'$ with $\bar{a}=-0{,}3$ (sinks); 4) (u/u_{m0}) with $\bar{a}=0{,}3$ (sources).

Fig. 6.5 shows the function $F_1'(\varphi)$, found in this way and also the function $F_0'(\varphi)$, corresponding to the impermeable plate. We can now write for the longitudinal velocity component in a first approximation

$$u=\frac{1}{2}\left(\frac{L}{\nu x}\right)^{1/2}\left[F_0'(\varphi)+ + q_0\left(\frac{x^3}{4\nu L}\right)^{1/4}F'(\varphi)+\ldots\right]. \tag{6.86}$$

As in paper [19] mentioned above, it can be proved that with finite coefficients Series (6.86) will be convergent on condition that the parameter $|\bar{a}|=|q_0\left(\frac{x^3}{4\nu L}\right)^{1/4}|<1$.

The calculated distributions of the nondimensional velocity $\frac{u}{u_{m0}}$, where $u_{m0}=\frac{1}{2}\left(\frac{L}{\nu x}\right)^{1/2}$ — is the maximum velocity in a jet cross section with neither sources nor sinks, are also shown in Fig. 6.5. We see from the diagram that with $\bar{a}>0$ (sources of fluid) the peak in the velocity

distribution is farther away from the plate and the effective thickness of the jet boundary layer is greater. In this case, the frictional stress at the plate is lower as follows from the expression

$$\tau_w = \mu \frac{\partial u}{\partial y}\Big|_{y=0} = \frac{\rho}{4x}\left(\frac{L^3}{4\nu x}\right)\left[F_0''(0) + \bar{a}F_1''(0) + \ldots\right]. \qquad (6.87)$$

From the condition $\tau_w = 0$ we find the separation point

$$x_{отр} = \left\{\frac{4\nu L}{q_0^4}\left[\frac{F_0''(0)}{F_1''(0)}\right]^4\right\}^{1/3} = \frac{0{,}089}{q_0}\left(\frac{4\nu L}{q_0}\right)^{1/3}. \qquad (6.88)$$

It follows from Eq. (6.88) that the position of the point of separation is determined by the properties of the fluid, the velocity of efflux of the jet and depends strongly on the injection velocity. The point of separation approaches the source as q_0 increases.

If $\bar{a} < 0$ (sinks of luid) the peak in the velocity distribution is closer to the plate, the jet becomes thinner and the frictional stress grows. According to the laws of variation of friction, we have the following expression for the relative flow rate of fluid in the jet (G_0 is the flow rate for an impermeable plate):

$$\frac{G}{G_0} = 1 + q_0\left(\frac{x^3}{4\nu L}\right)^{1/4}\frac{F_1(\infty)}{F_0(\infty)}. \qquad (6.89)$$

From the latter we see that if the plate contains sources ($q_0 > 0$) the flow rate increases and when it contains sinks ($q_0 < 0$) it drops.

The results obtained illustrate the influence of sources and sinks on the expansion of a semilimited jet.

Note that Eq. (6.78) is easy to integrate even for other values of the constant κ. For example, with $\varkappa = -1$ it can be reduced to a second-order equation

$$F'' + F_0F' + 5F_0'F = 4F_0'. \qquad (6.90)$$

One of the particular solutions corresponds to the well-known homogeneous equation: $F = F_0'$. Consequently, with $\varkappa = -1$ the problem can be solved by quadrature. Also easily integrable is the equation

$$F_1''' + F_0 F_1'' + 4F_0' F_1' + F_0'' F_1 = 4F_0'', \qquad (6.91)$$

which corresponds to the particular value of $\varkappa = -\frac{3}{4}$, as for the corresponding homogenous equation, two particular solutions can be given

$$F_1 = F_0' \text{ и } F_1 = F_0 + \varphi F_0'.$$

So we see that also here the solution can be reduced to a quadrature and the first-approximation solution obtained is the self-similar one.

The thermal problem of the expansion of a jet along a porous plate can be treated analogously.

Literature references

14, 17, 19, 53, 54, 71, 114, 170, 198, 202, 246.

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132	See footnote on page 83.

Chapter 7

LAMINAR JETS OF INCOMPRESSIBLE FLUID

7.1 THE SELF-SIMILAR SOLUTIONS

Most of the solutions of problems considered above belong to the class of self-similar jet motions caused by the efflux of various forms of source jets. It is a general feature of them that they can be treated within the framework of the theory of the asymptotic laminar boundary layer of an incompressible fluid. In this connection, it is expedient to follow Paper [57] and analyze the conditions at which a flow of the type of a source jet is self-similar in a general form. For this purpose, we again return to the equations of a free, laminar (plane or axisymmetric) boundary layer, (4.3). For simplicity, we assume the peripheral velocity component w equal to zero*.

In this case Eqs. (4.3) for the laminar problem take the form

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{1}{y^k}\frac{\partial}{\partial y}\left(y^k\frac{\partial u}{\partial y}\right), \tag{7.1}$$

$$\frac{\partial}{\partial x}(y^k u)+\frac{\partial}{\partial y}(y^k v)=0 \tag{7.2}$$

($k = 0$ corresponds to a plane and $k = 1$ to an axisymmetric boundary layer.

We admit that the two partial differential equations (7.1) and (7.2) can be reduced to one ordinary differential equation. For this purpose we set up a generalized self-similarity transformation:

$$\frac{u}{u_m}=\frac{F'(\varphi)}{\varphi^k},\quad u_m=f(x),\quad \varphi=y\cdot X(x). \tag{7.3}$$

For the sake of completeness of the calculation, we give the expres-

sions for all quantities entering Eqs. (7.1) and (7.2):

$$u = f(x)\frac{F'(\varphi)}{\varphi^k},$$

$$\frac{\partial u}{\partial x} = f'\frac{F'}{\varphi^k} + f\frac{X'}{X}\varphi\left(\frac{F'}{\varphi^k}\right)',$$

$$v = -\frac{1}{X\varphi^k}\left\{f'F + f\frac{X'}{X}\left[\varphi F' - (k+1)F\right]\right\},$$

$$\frac{\partial u}{\partial y} = fX\left(\frac{F'}{\varphi^k}\right)', \quad \frac{1}{y^k}\frac{\partial}{\partial y}\left(y^k\frac{\partial u}{\partial y}\right) = \frac{fX^2}{\varphi^k}\left[\varphi^k\left(\frac{F'}{\varphi^k}\right)'\right]'$$

(the primes indicate differentiation with respect to the "proper" variable).

We substitute these expressions in the initial equation (7.1) and investigate the conditions at which it can be reduced to an ordinary differential equation. As we see from the equation

$$ff'\left[\frac{F'^2}{\varphi^{2k}} - \frac{F}{\varphi^k}\left(\frac{F'}{\varphi^k}\right)'\right] + (k+1)f^2\frac{X'}{X}\frac{F}{\varphi^k}\left(\frac{F'}{\varphi^k}\right)' = \nu\frac{fX^2}{\varphi^k}\left[\varphi^k\left(\frac{F'}{\varphi^k}\right)'\right]', \quad (7.4)$$

these conditions consist in a proportionality of the terms which stand as coefficients in front of the functions which depend on the variable φ alone. In order words, we must put ($\sim$ the sign of proportionality)

$$f' \sim f\frac{X'}{X} \sim X^2.$$

These conditions are general for plane and axisymmetric flows.

From the first relation $\frac{f'}{f} \sim \frac{X'}{X}$ it follows that

$$f = \mathrm{const}\, X^n, \quad (7.5)$$

such that, taking (7.5) into account, we obtain from the second

$$X^{n-3}X' = \mathrm{const}. \quad (7.6)$$

In the case where the constant n is different from two, we obtain

$$\frac{X^{n-2}}{n-2} = \mathrm{const}\, x,$$

$$f = \mathrm{const}\,[(n-2)x]^{\frac{n}{n-2}}.$$

(The factor $n - 2$ has not been included in the const in order to stress the inequality $n \neq 2$).

Thus, with $n \neq 2$, the self-similarity transformation (7.3) takes

the form of power functions

$$u = \mathrm{const}\, x^{\alpha} \frac{F'}{\varphi^k}, \quad \varphi = \mathrm{const}\, y x^{\beta},$$

with the constants of self-similarity

$$\alpha = \frac{n}{n-2}, \quad \beta = \frac{1}{n-2}.$$

The concrete values of α and β can be determined with the help of an integral condition. For free jets this condition (momentum conservation) has the form

$$J_x = 2\pi^k \int_0^\infty \rho u^2 y^k \, dy = \mathrm{const}.$$

It permits the determination of the constant n:

$$J_x = 2\pi^k \rho \frac{f^2}{X^{k+1}} \int_0^\infty \frac{F'^2}{\varphi^k} d\varphi = \mathrm{const} \quad [X(x)^{2n-k-1} = \mathrm{const}].$$

This equation is possible if

$$n = \frac{k+1}{2}, \quad \alpha = \frac{k+1}{k-3}, \quad \beta = \frac{2}{k-3}. \tag{7.7}$$

In precisely this (exponential) form, with the values of the constants of self-similarity obtained here, the transformations of Type (7.3) were used above when solving problems on the expansion of free laminar souce jets (plane jet with $k = 0$ and axisymmetric jet with $k = 1$).

In the case of semilimited jets (to consider a concrete case, we only deal with the plane jet) the integral condition of the form

$$P = \int_0^\infty u^{3/2} dy = \mathrm{const}$$

yields $f^{3/2} = \mathrm{const}\, X$, and hence follows $n = \frac{2}{3}$, $\alpha = -\frac{1}{2}$, $\beta = -\frac{3}{4}$. These values also agree with those derived above.

Analogously, making use of Transformations (7.3) with $n \neq 2$ we also can consider the other self-similar jet flows discussed above (the jet boundary, at $\alpha = 0$ and $k = 0$, the free and the semilimited

fan-type jets, etc.). Moreover, as will be shown in the following, the general power laws of self-similarity can also be extended to self-similar turbulent jet flows of incompressible fluid.

Besides the exponential transformation formular corresponding to the inequality $n \neq 2$ we can in principle also apply self-similarity transformations of the form e^{mx} to jet flows. They correspond to a value of $n = 2$ in Eq. (7.6).

With $n = 2$ we have

$$\frac{X'}{X} = \text{const}, \quad X = \text{const}\, e^{mx}$$

and, taking (7.5) into account,

$$f = \text{const}\ e^{2mx},$$

where, owing to physical considerations (the limitedness of the speed of flow), we must assume that $m < 0$.

The transformations of self-similarity take the form

$$u = \text{const}\ e^{2mx} \frac{F'(\varphi)}{\varphi^k}, \quad \varphi = \text{const}\ y e^{mx}.$$

These transformation formulas do not satisfy the integral conditions (for the free or semilimited jets) and have not been applied in the theory of jet motions.

Similarly as with the dynamic problem, it is also easy to determine the conditions of self-similarity in the thermal (diffusion) problem [57]. The results which, for brevity, are not given here, are included in the summary table 7.1 in the following section.

7.2 THE FUNDAMENTAL RESULTS

Let us compare the results of solution of various problems of self-similar flows. Besides the local characteristics, we also consider some integral features to which, in general, the laws of variation along the jet belong for the following characteristics:

$G=\int\limits_0 \rho u\,ds$ the mass flow per second,

$J_x=\int\limits_0 \rho u^2 ds$ the momentum flux,

$M=\int\limits_0 \rho ruw\,ds$ the angular momentum flux,

$E=\frac{1}{2}\int\limits_0 \rho u^3 ds$ the kinetic energy flux,

$Q=\int\limits_0 \rho c_p u\,\Delta T\,ds$ the flux of excessive heat content,

$K=\int\limits_0 \rho u^2\left(\int\limits_0 \rho u\,ds\right)ds$ the first integral invariant of the semilimited jets,

$N=\int\limits_0 \rho ruw\left(\int\limits_0 \rho u\,ds\right)ds$ the second integral invariant of the semilimited jets,

$K_T=\int\limits_0 \Delta T^{1/2}\,ds$ the third (thermal) integral invariant of self-similar semilimited jets,

$K_T^1=\int\limits_0 \rho c_p u\,\Delta T\left(\int\limits_0 \rho u\,ds\right)ds$ the same for the case of Pr = 1.

To write these expressions as simply as possible, we used the denotation $ds=(2\pi r)^k dy$ for the areal element of a jet cross section where the distance to the axis of symmetry $r \equiv y$ in problems on the expansion of jets with Ox being the axis of symmetry directed along the flow and $r \equiv x$ for jets with Oy as the axis of symmetry (in the case of a conic jet $r \equiv x \sin\omega$, where 2ω is the vertex angle of the cone).

We have compiled the results of solution of various problems for the purpose of convenient comparison and application in a single summary table for self-similar solutions to problems on the expansion of laminar jets of incompressible fluids. In Table 7.1 (see below) for each solution to the dynamic problem and the pertinent variants of solutions to the thermal problem (for different boundary conditions for the temperature) we given, in this order, the initial equations, the boundary conditions, and so on, up to the integral laws.

It must be mentioned that the results of solution of self-similar equations contained in this table will be used in what follows (with other values of the constants of self-similarity) in the theory

and calculation of turbulent jets of incompressible fluids.

As already mentioned, this summary table does not contain the results of approximate solutions of problems on the propagation of a jet in a parallel (or antiparallel) uniform flow of the problems considered in the previous three chapters within the framework of the theory of the laminar, free and semilimited jet boundary layers, since these solutions, which were obtained by the method of small perturbations and taking into account the "initial" conditions of efflux, do not contain to the self-similar solutions.

Before we pass over to laminar jets of compressible gas, we consider yet another example of jet flow, namely the formation of a peculiar boundary layer at the interface of two collinear flows of fluids which do not mix.

TABLE 7.1

Self-Similar Laminar Jets of Incompressible Fluids

Scheme of Table

1) Form of flow;
2) differential equations;
3) boundary conditions;
4) transformation formulas:
5) constants of self-similarity;
6) constants in the transformation formulas;
7) integral conditions of conservation:
8) self-similar equations;
9) boundary conditions;
10) solutions;
11) integral characteristics.

Remarks

In lines 2, 3, etc., *I* refers to the dynamic problem, *II* to the thermal (diffusion) problem.

In the lines 3, 4, etc. *IIa* refers to the thermal problem with symmetrical boundary conditions, *IIb* to it with asymmetrical boundary conditions (for the temperature).

In the tables for semilimited jets:

IIa refers to the thermal problem with $T_w = T_\infty$, *IIb* to the thermal problem with a nonheatconducting wall, *IIc* to the thermal problem with T_w = const T_∞.

1		A Свободная плоская струя		
2	I	$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$; $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$		
	II	$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = a\frac{\partial^2 T}{\partial y^2}$		
3	I	$v(x, 0) = \frac{\partial u}{\partial y}\Big\vert_{y=0} = 0$; $u(x, \pm\infty) = 0$		
	IIa	$\frac{\partial T}{\partial y}\Big\vert_{y=0} = 0$, $T(x, \pm\infty) = T_\infty$	IIб	$T(x, +\infty) = T_1$, $T(x, -\infty) = T_2$

TABLE 7.1 continued

1		A Свободная плоская струя		
4	I	$\frac{u}{u_m} = F'(\varphi), \quad u_m = Ax^{\alpha}, \quad \varphi = Byx^{\beta}$		
	IIa	$\frac{T - T_\infty}{T_m - T_\infty} = \theta(\varphi), \quad T_m - T_\infty = \Gamma x^{\gamma}$	IIб	$\frac{T - T_2}{T_1 - T_2} = \theta(\varphi)$
5	I	$\alpha = -\frac{1}{3}, \quad \beta = -\frac{2}{3}$		
	IIa	$\gamma = -\frac{1}{3}$	IIб	$(\gamma = 0)$
6	I	$A = \frac{1}{2}\sqrt[3]{\frac{3J_x^2}{4\rho^2\nu}}, \quad B = \frac{1}{2}\sqrt[3]{\frac{J_x}{6\rho\nu^2}}$		
	IIa	$\Gamma = \frac{Q}{c_p}\sqrt[3]{\frac{2}{9\rho^2\nu J_x}}\left[\int\limits_{-\infty}^{+\infty}(F')^{Pr+1}d\varphi\right]^{-1}$	IIб	$(\Gamma = T_1 - T_2)$
7	I	$J_x = \int\limits_{-\infty}^{+\infty}\rho u^2\, dy = \mathrm{const}$		
	IIa	$Q = \int\limits_{-\infty}^{+\infty}\rho c_p u\,(T - T_\infty)\, dy = \mathrm{const}$	IIб	—
8	I	$F''' + 2(FF'' + F'^2) = 0$		
	IIa	$\theta'' + 2Pr\,(F\theta' + F'\theta) = 0$	IIб	$\theta'' + 2PrF\theta' = 0$
9	I	$F(0) = 0, \quad F'(0) = 1, \quad F'(\pm\infty) = 0$		
	IIa	$\theta'(0) = 0, \quad \theta(\pm\infty) = 0$	IIб	$\theta(+\infty) = 1, \quad \theta(-\infty) = 0$
10	I	$F = \mathrm{th}\,\varphi, \quad F' = 1 - \mathrm{th}^2\varphi$		
	IIa	$\theta(\varphi) = (F')^{Pr} = (1 - \mathrm{th}^2\varphi)^{Pr}$	IIб	$\theta(\varphi) = \left[\int\limits_{-\infty}^{\varphi}(\mathrm{ch}\,\varphi)^{-2Pr}\, d\varphi\right]\left[\int\limits_{-\infty}^{+\infty}(\mathrm{ch}\,\varphi)^{-2Pr}\, d\varphi\right]^{-1}$
11	I	$G = \sqrt[3]{36\rho^2\nu J_x x}, \quad E = \frac{J_x}{30\mu}\sqrt[3]{6\rho\nu^2 J_x^2\frac{1}{x}}$		
	IIa	—	IIб	$Q = \frac{1}{2}c_p(T_1 - T_2)\left[\int\limits_{-\infty}^{+\infty}F'\theta\, d\varphi\right]\sqrt[3]{36\rho^2\nu J_x}$

TABLE 7.1 continued

1		C Спутные или встречные потоки		
2	I	$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$; $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$		
	II	$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = a\frac{\partial^2 T}{\partial y^2}$		
3	I	$u(x, +\infty) = u_1$ $\left.\frac{\partial u}{\partial y}\right\|_{y=\pm\infty} = 0$; $u(x, -\infty) = u_2$		
	IIa	—	IIб	$T(x, +\infty) = T_1$, $T(x, -\infty) = T_2$
4	I	$\frac{u}{u_m} = F'(\varphi)$, $u_m = u_1 = \mathrm{const}$, $\varphi = Byx^{\beta}$		
	IIa	—	IIб	$\frac{T - T_2}{T_m - T_2} = \theta(\varphi)$, $T_m = T_1 = \mathrm{const}$, $T_2 = \mathrm{const}$
5	I	$\alpha = 0$, $\beta = -\frac{1}{2}$		
	IIa	—	IIб	$\gamma = 0$
6	I	$(A = u_1)$, $B = \frac{1}{2}\sqrt{\frac{u_1}{\nu}}$		
	IIa	—	IIб	$(\Gamma = T_1 - T_2)$
7	I	—		
	IIa	—	IIб	—
8	I	$F''' + 2FF'' = 0$		
	IIa	—	IIб	$\theta'' + 2\mathrm{Pr}F\theta' = 0$]
9	I	$F'(+\infty) = 1$, $F''(+\infty) = 0$; $F'(-\infty) = m = \frac{u_2}{u_1}$, $F''(-\infty) = 0$		
	IIa	—	IIб	$\theta(+\infty) = 1$, $\theta(-\infty) = 0$
10	I	$\frac{u}{u_1} = F'(\varphi) = 1 + \frac{1}{2}(m-1)(1 - \mathrm{erf}\,\varphi)$ $\left(\mathrm{erf}\,\varphi = \frac{2}{\sqrt{\pi}}\int_0^{\varphi} e^{-t^2}dt\right)$		
	IIa	—	IIб	$\theta = \frac{T - T_2}{T_1 - T_2} = \frac{1}{2}[1 + \mathrm{erf}(\varphi\sqrt{\mathrm{Pr}})]$
11	I	—		
	IIa	—	IIб	—

TABLE 7.1 continued

1		D Свободная осесимметричная струя		
2	I	$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial y}\right);\ \frac{\rho w^2}{y}=\frac{\partial p}{\partial y};\ u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+\frac{vw}{y}=\nu\left\{\frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial w}{\partial y}\right)-\frac{w}{y^2}\right\};\ \frac{\partial}{\partial x}(yu)+\frac{\partial}{\partial y}(yv)=0$		
	II	$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=a\frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial T}{\partial y}\right)$		
3	I	$v(x,0)=w(x,0)=\left.\frac{\partial u}{\partial y}\right\vert_{y=0}=\left.\frac{\partial w}{\partial y}\right\vert_{y=0}=0,$ $u(x,\infty)=w(x,\infty)=0,\ \ p(x,\infty)=p_\infty$		
	IIa	$\left.\frac{\partial T}{\partial y}\right\vert_{y=0}=0,\ \ T(x,\infty)=T_\infty$	IIб	—
4	I	$\frac{u}{u_m}=\frac{F'(\varphi)}{\varphi},\ \frac{w}{w_m}=\Phi(\varphi),\ \frac{p-p_\infty}{p_m-p_\infty}=P(\varphi);\ u_m=Ax^\alpha,\ w_m=Cx^\varepsilon,\ p_m-p_\infty=\rho Dx^\delta,\ \varphi=Byx^\beta$		
	IIa	$\frac{T-T_\infty}{T_m-T_\infty}=\theta(\varphi),\ \ T_m-T_\infty=\Gamma x^\gamma$	IIб	—
5	I	$\alpha=-1,\ \beta=-1,\ \varepsilon=-2,\ \delta=-4$		
	IIa	$\gamma=-1$	IIб	—
6	I	$A=\frac{3J_x}{8\pi\rho\nu},\ B=\sqrt{\frac{3J_x}{8\pi\rho\nu^2}},\ C=\frac{3M_x}{32\pi\rho\nu^2}\sqrt{\frac{3J_x}{8\pi\rho}},\ D=\frac{9}{2048}\frac{M_x^2J_x}{\pi^3\rho^3\nu^4}$		
	IIa	$\Gamma=\frac{(2Pr+1)\,Q}{8\pi\rho\nu c_p}$	IIб	—
7	I	$J_x=2\pi\int_0^\infty\rho u^2y\,dy=\text{const},\ \ M_x=2\pi\int_0^\infty\rho uwy^2\,dy=\text{const}$		
	IIa	$Q=2\pi\int_0^\infty\rho c_pu(T-T_\infty)\,y\,dy=\text{const}$	IIб	—
8	I	$\left(F''-\frac{F'}{\varphi}\right)'+\left(\frac{FF'}{\varphi}\right)'=0;\ \ P'=\frac{\Phi^2}{\varphi};\ \ \Phi''+\frac{1+F}{\varphi}\Phi'+\frac{\varphi F'+F-1}{\varphi^2}\Phi=0$		
	IIa	$(\varphi\theta')'+Pr\,(F\theta)'=0$	IIб	—
9	I	$\left.\frac{F'}{\varphi}\right\vert_{\varphi=0}=1,\ \left.\frac{F}{\varphi}\right\vert_{\varphi=0}=0,\ \ \Phi(0)=0;\ \left.\frac{F'}{\varphi}\right\vert_{\varphi=\infty}=0,\ \ \Phi(\infty)=0;\ \ P(\infty)=0$		
	IIa	$\theta'(0)=0,\ \ \theta(\infty)=0$	IIб	—
10	I	$F(\varphi)=\frac{\frac{1}{2}\varphi^2}{1+\frac{1}{8}\varphi^2},\ \ \frac{F'}{\varphi}=\frac{1}{\left(1+\frac{1}{8}\varphi^2\right)^2},\ \ \Phi=\frac{\varphi}{\left(1+\frac{1}{8}\varphi^2\right)^2},\ \ P(\varphi)=\frac{1}{\left(1+\frac{1}{8}\varphi^2\right)^4}$		
	IIa	$\theta(\varphi)=\left(\frac{F'}{\varphi}\right)^{Pr}=\frac{1}{\left(1+\frac{1}{8}\varphi^2\right)^{2Pr}}$	IIб	—

TABLE 7.1 continued

1		D Свободная осесимметричная струя		
11	I	$G=\int\limits_0^\infty 2\pi\rho u y\,dy = 8\pi\mu x;\ E=\pi\int\limits_0^\infty \rho u^3 y\,dy = \frac{\pi}{\mu}\left(\frac{3J_x}{8\pi}\right)^2\left[\int\limits_0^\infty \frac{F'^3}{\varphi^2}\,d\varphi\right]\frac{1}{x}$		
	IIa	—	IIб	—

1		E Свободная веерная струя		
2	I	$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2};\ u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+\frac{uw}{x}=\nu\frac{\partial^2 w}{\partial y^2};\ \frac{\partial}{\partial x}(xu)+\frac{\partial}{\partial y}(xv)=0$		
	II	$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=a\frac{\partial^2 T}{\partial y^2}$		
3	I	$v(x,0)=0,\ \frac{\partial u}{\partial y}\Big\vert_{y=0}=\frac{\partial w}{\partial y}\Big\vert_{y=0}=0;\ u(x,\pm\infty)=w(x,\pm\infty)=0$		
	IIa	$\frac{\partial T}{\partial y}\Big\vert_{y=0}=0,\ \ T(x,\pm\infty)=T_\infty$	IIб	$T(x,+\infty)=T_1,$ $T(x,-\infty)=T_2$
4	I	$\frac{u}{u_m}=F'(\varphi),\ \frac{w}{w_m}=\Phi(\varphi);\ u_m=Ax^\alpha,\ \ w_m=Cx^\varepsilon,\ \ \varphi=Byx^\beta$		
	IIa	$\frac{T-T_\infty}{T_m-T_\infty}=\theta(\varphi),\ \ T_m-T_\infty=\Gamma x^\gamma$	IIб	$\frac{T-T_2}{T_1-T_2}=\theta(\varphi)$
5	I	$\alpha=-1,\ \ \beta=-1,\ \varepsilon=-2$		
	IIa	$\gamma=-1$	IIб	$(\gamma=0)$
6	I	$A=\frac{1}{4}\sqrt[3]{\frac{9J_x^2}{2\pi^2\rho^2\nu}},\ \ B=\frac{1}{2}\sqrt[3]{\frac{3J_x}{4\pi\rho\nu^2}},\ \ C=\frac{3M_y}{8\pi\rho}\sqrt[3]{\frac{4\pi\rho}{3\nu J_x}}$		
	IIa	$\Gamma=\frac{Q}{2\pi\rho c_p J_2}\sqrt[3]{\frac{4\pi\rho}{3\nu J_x}}\ \left(J_2=\int\limits_{-\infty}^{+\infty}F'\theta\,d\varphi\right)$	IIб	$(\Gamma=T_1-T_2)$
7	I	$J_x=2\pi x\int\limits_{-\infty}^{+\infty}\rho u^2\,dy=\text{const},\ \ M_y=2\pi x^2\int\limits_{-\infty}^{+\infty}\rho uw\ dy=\text{const}$		
	IIa	$Q=2\pi x\int\limits_{-\infty}^{+\infty}\rho c_p u(T-T_\infty)\,dy=\text{const}$	IIб	—
8	I	$F'''+2(FF')'=0,\ \ \ \Phi''+2(F\Phi)'=0$		
	IIa	$\theta''+2Pr(F\theta)'=0$	IIб	$\theta''+2PrF\theta'=0$
9	I	$F(0)=0,\ \ F'(0)=1,\ \ F'(\pm\infty)=\Phi(\pm\infty)=0;\ \ \Phi'(0)=0$		
	IIa	$\theta'(0)=0,\ \ \theta(\pm\infty)=0$	IIб	$\theta(+\infty)=1,\ \ \theta(-\infty)=0$

TABLE 7.1 continued

1		E Свободная веерная струя				
10	I	$F = \mathrm{th}\,\varphi,\quad F' = 1-\mathrm{th}^2\varphi;\quad \Phi = F' = 1-\mathrm{th}^2\varphi$				
	IIa	$\theta(\varphi) = (F')^{Pr} = (\mathrm{ch}\,\varphi)^{-2Pr}$	IIб	$\theta(\varphi) = \left[\int_{-\infty}^{\varphi}(\mathrm{ch}\,\varphi)^{-2Pr}\,d\varphi\right]\left[\int_{-\infty}^{+\infty}(\mathrm{ch}\,\varphi)^{-2Pr}\,d\varphi\right]^{-1}$		
11	I	$G = 2\pi x\int_{-\infty}^{+\infty}\rho u\,dy = 2\sqrt[3]{6\pi^2\rho^2\nu J_x x},\quad E = \pi x\int_{-\infty}^{+\infty}\rho u^3\,dy = \frac{3}{20}\frac{J_x}{\pi\mu}\sqrt[3]{\frac{4}{3}\pi\rho\nu^2 J_x^2}\,\frac{1}{x}$				
	IIa	—	IIб	$Q = c_p(T_1-T_2)\sqrt[3]{6\pi^2\rho^2\nu J_x x}\int_{-\infty}^{+\infty}F'\theta\,d\varphi$		

1		F Плоская полуограниченная струя				
2	I	$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2};\quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$				
	II	$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} = a\frac{\partial^2 T}{\partial y^2}$				
3	I	$u(x,0)=v(x,0)=0;\quad u(x,\infty)=0$				
	IIa	$T(x,0)=T_\infty,\ T(x,\infty)=T_\infty$	IIб	$\left.\frac{\partial T}{\partial y}\right\|_{y=0}=0,\ T(x,\infty)=T_\infty$	IIв	$T(x,0)=T_w,\ T(x,\infty)=T_\infty$
4	I	$\frac{u}{u_m}=F'(\varphi),\quad u_m = Ax^{\alpha},\quad \varphi = Byx^{\beta}$				
	IIa	$\frac{T-T_\infty}{T_m-T_\infty}=\theta(\varphi),\ T_m-T_\infty=\Gamma x^{\gamma}$	IIб	$\frac{T-T_\infty}{T_m-T_\infty}=\theta(\varphi),\ T_m-T_\infty=\Gamma x^{\gamma}$	IIв	$\frac{T-T_\infty}{T_w-T_\infty}=\theta(\varphi)\quad(\Gamma = T_w-T_\infty)$
5	I	$\alpha=-\frac{1}{2},\quad \beta=-\frac{3}{4}$				
	IIa	$\gamma=-\frac{1}{2}$	IIб	$\gamma=-\frac{1}{4}$	IIв	$(\gamma=0)$
6	I	$A=\sqrt{\frac{K}{4\rho^2\nu J_1}},\quad B=\frac{1}{2}\sqrt[4]{\frac{K}{4\rho^2\nu^3 J_1}}\quad\left(J_1=\int_0^\infty FF'\,d\varphi\right)$				
	IIa	$\Gamma(Pr)=\left[\int_0^\infty\theta^{3/2}d\varphi\,(K_T B)^{-1}\right]^{2/3}$	IIб	$\Gamma=\frac{Q}{c_p}\sqrt[4]{\frac{J_1}{4\rho^2\nu K}}\left[\int_0^\infty F'\theta\,d\varphi\right]^{-1}$	IIв	$(\Gamma = T_w-T_\infty)$
7	I	$K=\int_0^\infty\rho u^2\left(\int_0^y\rho u\,dy\right)dy=\mathrm{const}$				
	IIa	$K_T=\int_0^\infty(T-T_\infty)^{3/2}dy=\mathrm{const}$	IIб	$Q=\int_0^\infty\rho c_p u\,(T-T_\infty)\,dy=\mathrm{const}$	IIв	—

TABLE 7.1 continued

1		F Плоская полуограниченная струя				
8	I	$F''' + FF'' + 2F'^2 = 0$				
	IIa	$\frac{1}{Pr}\theta'' + F\theta' + 2F'\theta = 0$	IIб	$\theta'' + Pr(F\theta)' = 0$	IIв	$\theta'' + PrF\theta' = 0$
9	I	$F(0) = F'(0) = 0;\quad F'(\infty) = 0$				
	IIa	$\theta(0) = 0,\quad \theta(\infty) = 0$	IIб	$\theta'(0) = 0,\quad \theta(\infty) = 0$	IIв	$\theta(0) = 1,\quad \theta(\infty) = 0$
10	I	$\varphi = \frac{1}{2F_\infty}\ln\frac{F + \sqrt{FF_\infty} + F_\infty}{(\sqrt{F_\infty} - \sqrt{F})^2} + \frac{\sqrt{3}}{F_\infty}\left(\mathrm{arctg}\,\frac{2\sqrt{F} + \sqrt{F_\infty}}{\sqrt{3F_\infty}} - \mathrm{arctg}\,\frac{1}{\sqrt{3}}\right);$ $F' = \frac{2}{3}(F_\infty^{3/2}F^{1/2} - F^2);\quad F(\infty) = 1{,}7818$				
	IIa	$\theta(\varphi)_{Pr=1} = F'(\varphi)$	IIб	$\theta(\varphi) = \exp\left(-Pr\int_0^\varphi F\,d\varphi\right)$	IIв	$\theta(\varphi) = 1 -$ $-\left[\int_0^\varphi \exp\left(-Pr\int_0^\varphi F\,d\varphi\right)d\varphi\right]\times$ $\times\left[\int_0^\infty \exp\left(-Pr\int_0^\infty F\,d\varphi\right)d\varphi\right]^{-1}$
11	I	$G = \int_0^\infty \rho u\,dy = F_\infty\sqrt[4]{\frac{4\rho^2\nu Kx}{J_1}};\quad J_x = \sqrt[4]{\frac{K^3}{4\rho^2\nu J_1^3 x}}\left[\int_0^\infty F'^2\,d\varphi\right];\quad E = \frac{K}{4\rho J_1}\sqrt[4]{\frac{K}{4\rho^2\nu^3 J_1 x^3}}$				
	IIa	$Q = \rho c_p\frac{A\Gamma}{B}x^{-1/4}\int_0^\infty F'\theta\,d\varphi$	IIб	—	IIв	$Q = \sqrt[4]{\frac{4\rho^2\nu Kx}{J_1}}\,c_p(T_w - T_\infty)\times$ $\times\int_0^\infty F'\theta\,d\varphi$

TABLE 7.1 continued

1		G Слабозакрученная струя, распространяющаяся вдоль конуса				
2	I	$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$; $\rho\frac{w^2}{x}\operatorname{ctg}\omega=\frac{\partial p}{\partial y}$; $u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+\frac{uw}{x}=\nu\frac{\partial^2 w}{\partial y^2}$; $\frac{\partial}{\partial x}(xu)+\frac{\partial}{\partial y}(xv)=0$				
	II	$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=a\frac{\partial^2 T}{\partial y^2}$				
3	I	$u(x,0)=v(x,0)=w(x,0)=0,\quad u(x,\infty)=w(x,\infty)=0,\quad p(x,\infty)=p_\infty$				
	IIa	$T(x,0)=T_\infty$, $T(x,\infty)=T_\infty$	IIб	$\left.\frac{\partial T}{\partial y}\right\vert_{y=0}=0,\quad T(x,\infty)=T_\infty$	IIв	$T(x,0)=T_w$, $T(x,\infty)=T_\infty$
4	I	$\frac{u}{u_m}=F'(\varphi)$, $\frac{w}{w_m}=\Phi(\varphi)$, $\frac{p_\infty-p}{p_\infty-p_m}=P(\varphi)$; $u_m=Ax^\alpha$, $w_m=Cx^\varepsilon$, $p_\infty-p_m=Dx^\delta$, $\varphi=Byx^\beta$				
	IIa	$\frac{T-T_\infty}{T_m-T_\infty}=\theta(\varphi)$, $T_m-T_\infty=\Gamma x^\gamma$	IIб	$\frac{T-T_\infty}{T_m-T_\infty}=\theta(\varphi)$, $T_m-T_\infty=\Gamma x^\gamma$	IIв	$\frac{T-T_\infty}{T_w-T_\infty}=\theta(\varphi)$, $(\Gamma=T_w-T_\infty)$
5	I	$\alpha=-\frac{3}{2}$, $\beta=-\frac{5}{4}$, $\varepsilon=-\frac{5}{2}$, $\delta=-\frac{19}{4}$				
	IIa	$\gamma=-\frac{3}{2}$	IIб	$\gamma=-\frac{3}{4}$	IIв	$(\gamma=0)$
6	I	$A=\sqrt{\frac{3K}{4\rho^2\nu J_1}}$, $B=\sqrt[4]{\frac{3^3K}{4^3\rho^2\nu^3J_1}}$, $C=\frac{N}{\rho}\sqrt{\frac{3}{4\nu KJ_1}}$, $D=\operatorname{ctg}\omega\left(\frac{N}{K}\right)^2\sqrt[4]{\frac{3}{4\rho^2\nu}\left(\frac{K}{J_1}\right)^3}$ $\left(J_1=\int_0^\infty FF'^2\,d\varphi\right)$				
	IIa	$\Gamma(Pr)=\left[\int_0^\infty\theta^{3/2}\,d\varphi(BK_T)^{-1}\right]^{-\frac{2}{3}}$	IIб	$\Gamma=\frac{Q}{2\pi\rho c_p\sin\omega}\sqrt[4]{\frac{3J_1}{4\rho^2\nu K}}\times\left[\int_0^\infty F'\theta\,d\varphi\right]^{-1}$	IIв	$\Gamma=T_w-T_\infty$
7	I	$K=\int_0^\infty\rho xu^2\left(\int_0^y\rho xu\,dy\right)dy=\mathrm{const}$, $N=\int_0^\infty\rho x^2uw\int_0^y\rho xu\,dy)\,dy=\mathrm{const}$				
	IIa	$K_T=\int_0^\infty(T-T_\infty)^{3/2}x\,dy=\mathrm{const}$	IIб	$Q=2\pi\sin\omega\int_0^\infty\rho xuc_p\times(T-T_\infty)\,dy=\mathrm{const}$	IIв	—
8	I	$F'''+FF''+2F'^2=0$; $P'=-\Phi^2$; $\Phi''+F\Phi'+2F'\Phi=0$				
	IIa	$\frac{1}{Pr}\theta''+F\theta'+2F'\theta=0$	IIб	$\theta''+Pr(F\theta)'=0$	IIв	$\theta''+PrF\theta'=0$
9	I	$F(0)=F'(0)=\Phi(0)=0$; $F'(\infty)=\Phi(\infty)=P(\infty)=0$				
	IIa	$\theta(0)=0,\quad\theta(\infty)=0$	IIб	$\theta'(0)=0,\quad\theta(\infty)=0$	IIв	$\theta(0)=1,\quad\theta(\infty)=0$

TABLE 7.1 continued

1		G Слабозакрученная струя, распространяющаяся вдоль конуса				
10	I	$\varphi = \frac{1}{2F_\infty}\left\{\ln\frac{F+\sqrt{FF_\infty}+F_\infty}{(\sqrt{F}-\sqrt{F_\infty})^2} + 2\sqrt{3}\left(\operatorname{arctg}\frac{2\sqrt{F}+\sqrt{F_\infty}}{\sqrt{3F_\infty}} - \operatorname{arctg}\frac{1}{\sqrt{3}}\right)\right\}$; $F' = \Phi = \frac{2}{3}(F_\infty^{3/2}F^{1/2} - F^2)$; $P = -\int\limits_\varphi^\infty F'^2\,d\varphi$				
	IIa	$\theta(\varphi)_{Pr=1} = F'(\varphi)$	IIб	$\theta(\varphi) = \exp\left(-Pr\int\limits_0^\varphi F\,d\varphi\right)$	IIв	$\theta(\varphi) = 1 - \left[\int\limits_0^\varphi \exp\left(-Pr\int\limits_0^\varphi F\,d\varphi\right)d\varphi\right] \times \left[\int\limits_0^\infty \exp\left(-\int\limits_0^\infty Pr\,F\,d\varphi\right)d\varphi\right]^{-1}$
11	I	$G = 2\pi\rho F_\infty \sin\omega\frac{A}{B}x^{3/4}$; $J_x = 2\pi\rho\sin\omega\frac{A^2}{B}x^{-1/4}\int\limits_0^\varphi F'^2\,d\varphi$; $M = 2\pi\rho\frac{AC}{B}\sin^2\omega x^{-1/4}\int\limits_0^\infty F'^2\,d\varphi$; $E = \pi\rho\sin\omega\frac{A^3}{B}x^{-1/4}\int\limits_0^\infty F'^3\,d\varphi$				
	IIa	$Q = 2\pi\rho c_p\frac{A\Gamma}{B}x^{-3/4}\sin\omega\cdot\int\limits_0^\infty F'\theta\,d\varphi$	IIб	—	IIв	$Q = 2\pi\rho c_p \times \sin\omega\frac{A}{B}(T_w - T_\infty)x^{3/4}\int\limits_0^\infty F'\theta\,d\varphi$

A) Free plane jet; C) parallel or anti-parallel flows; D) free axi-symmetric jet; E) free fan-type jet; F) plane semilimited jet; G) slightly twisted jet expanding along a cone.

7.3 THE FLOW AT THE INTERFACE OF IMMISCIBLE FLUIDS

Let us consider the problem of parallel and antiparallel laminar flows, analogously as in Section 5.3, but with the essential difference that the uniform flows are flows of immiscible fluids. The two uniform flows of fluid are assumed characterized by the values of velocity, temperature, density, the coefficients of viscosity, heat conduction and specific heat denoted by u_1, T_1, ρ_1, μ_1, λ_1, c_{p1} and u_2, T_2, ρ_2, μ_2, λ_2, c_{p2}, respectively.

At point O the two flows come into touch and each of the fluid forms boundary layers (dynamic and thermal) along the interface which coincides with the axis Ox. We investigate the flows in these boundary

layers under the assumption that the physical characteristics of the
f
fluids remain unchanged.

The equations of motion and heat transfer for each of the fluids in the range $x > 0$ can be written as follows:

$$\left.\begin{aligned}
&u^{(1)}\frac{\partial u^{(1)}}{\partial x}+v^{(1)}\frac{\partial u^{(1)}}{\partial y}=\nu_1\frac{\partial^2 u^{(1)}}{\partial y^2},\\
&\frac{\partial u^{(1)}}{\partial x}+\frac{\partial v^{(1)}}{\partial y}=0,\\
&u^{(1)}\frac{\partial T^{(1)}}{\partial x}+v^{(1)}\frac{\partial T^{(1)}}{\partial y}=a_1\frac{\partial^2 T^{(1)}}{\partial y^2}
\end{aligned}\right\}\tag{7.8}$$

and

$$\left.\begin{aligned}
&u^{(2)}\frac{\partial u^{(2)}}{\partial x}+v^{(2)}\frac{\partial u^{(2)}}{\partial y}=\nu_2\frac{\partial^2 u^{(2)}}{\partial y^2},\\
&\frac{\partial u^{(2)}}{\partial x}+\frac{\partial v^{(2)}}{\partial y}=0,\\
&u^{(2)}\frac{\partial T^{(2)}}{\partial x}+v^{(2)}\frac{\partial T^{(2)}}{\partial y}=a_2\frac{\partial^2 T^{(2)}}{\partial y^2}.
\end{aligned}\right\}\tag{7.9}$$

The systems of Eqs. (7.8) and (7.9) must be integrated under the following boundary conditions:

$$\left.\begin{aligned}
&u^{(1)}(x,+\infty)=u_1,\ T^{(1)}(x,+\infty)=T_1;\quad u^{(1)}(x,0)=u^{2}(x,0);\\
&u^{(2)}(x,-\infty)=u_2,\ T^{(2)}(x,-\infty)=T_2;\quad T^{(1)}(x,0)=T^{(2)}(x,0)\\
&\mu_1\frac{\partial u^{(1)}}{\partial y}\Big|_{y=0}=\mu_2\frac{\partial u^{(2)}}{\partial y}\Big|_{y=0},\ \lambda_1\frac{\partial T^{(1)}}{\partial y}\Big|_{y=0}=\lambda_2\frac{\partial T^{(2)}}{\partial y}\Big|_{y=0}
\end{aligned}\right\}\tag{7.10}$$

As it is rather cumbersome to solve all systems of Eqs. (7.8)-(7.9) we first consider the dynamic problem separately.

We suppose that

$$u^{(1)}=u_1F_1'(\varphi),\quad u^{(2)}=u_1F_2'(\varphi_2),\tag{7.11}$$

where

$$\varphi_1=\frac{y}{2}\sqrt{\frac{u_1}{\nu_1 x}},\quad \varphi_2=\frac{y}{2}\sqrt{\frac{u_1}{\nu_2 x}}.$$

In this case the problem can be reduced to integrating the ordinary aifferential equations

$$F_1'''(\varphi_1)+2F_1(\varphi_1)F_1''(\varphi_1)=0,\tag{7.12}$$

$$F_2'''(\varphi_2)+2F_2(\varphi_2)F_2''(\varphi_2)=0\tag{7.13}$$

with the following boundary conditions for the functions $F_1(\varphi_1)$ and $F_2(\varphi_2)$:

$$\left.\begin{aligned} F_1'(+\infty) = 1, \quad F_2'(-\infty) = \frac{u_2}{u_1} = m, \\ F_1'(0) = F_2'(0), \quad F_1''(0) = \Omega F_2''(0). \end{aligned}\right\} \tag{7.14}$$

$$\left(\text{where } \Omega = \sqrt{\frac{\rho_1\mu_1}{\rho_2\mu_2}}.\right)$$

A solution to Eqs. (7.12) and (7.13) can be found by, e.g., the method of successive approximations (similarly as this was done in Section 5.3) where we take as the zeroth approximation the functions

$$F_{10}(\varphi_1) = \varphi_1 \text{ and } F_{20}(\varphi_2) = \varphi_2.$$

For the functions of the first approximation we then obtain the equations

$$\left.\begin{aligned} F_1''' + 2\varphi_1 F_1'' = 0, \\ F_2''' + 2\varphi_2 F_2'' = 0, \end{aligned}\right\} \tag{7.15}$$

which are quite analogous to the first equation of System (5.26). General solutions of the latter equations can be written in the form of

$$\left.\begin{aligned} F_1(\varphi_1) = C_1 \int_0^{\varphi_1} (\operatorname{erf} z)\, dz + C_3\varphi + C_5, \\ F_2(\varphi_2) = C_2 \int_0^{\varphi_2} (\operatorname{erf} z)\, dz + C_4\varphi + C_6. \end{aligned}\right\} \tag{7.16}$$

Hence we obtain

$$\left.\begin{aligned} F_1'(\varphi_1) = C_1 \operatorname{erf}(\varphi_1) + C_3, \\ F_2'(\varphi_2) = C_2 \operatorname{erf}(\varphi_2) + C_4. \end{aligned}\right\} \tag{7.17}$$

Using the boundary conditions (7.10) we can determine the constants of integration C_1 to C_4. The simple calculations result in the following final expressions for the functions $F_1'(\varphi_1)$ and $F_2'(\varphi_2)$:

$$\left.\begin{aligned} F_1'(\varphi_1) = \frac{1}{1+\Omega}\{\Omega + m + (1-m)\operatorname{erf}\varphi_1\} = \frac{u^{(1)}}{u_1}, \\ F_2'(\varphi_2) = \frac{1}{1+\Omega}\{\Omega + m + \Omega(1-m)\operatorname{erf}\varphi_2\} = \frac{u^{(2)}}{u_1}. \end{aligned}\right\} \tag{7.18}$$

With $\Omega = 1$ (which corresponds to the mixing of two uniform flows of fluids of like physical properties) the two latter expressions coincide with the first expression of (5.30).

With the help of the equations

$$u^{(1)} = \frac{\partial\psi_1}{\partial y},$$
$$u^{(2)} = \frac{\partial\psi_2}{\partial y}$$

and Eq. (7.18) it is easy to obtain expressions for the streamfunctions in the ranges $y > 0$ and $y < 0$ (with $x > 0$):

$$\left.\begin{aligned} \frac{\psi_1}{2\sqrt{\nu_1 u_1 x}} &= \frac{1}{1+\Omega}\left\{(m+\Omega)\varphi_1 + (1-m)\left[\int_0^{\varphi_1}(\operatorname{erf} z)\,dz + C_7\right]\right\}, \\ \frac{\psi_2}{2\sqrt{\nu_2 u_1 x}} &= \frac{1}{1+\Omega}\left\{(m+\Omega)\varphi_2 + \Omega(1-m)\left[\int_0^{\varphi_2}(\operatorname{erf} z)\,dz + C_8\right]\right\}. \end{aligned}\right\} \quad (7.19)$$

If we take the axis Ox, i.e., the interface of the fluids, as the zero streamline of each of the flows, the constants C_7 and C_8 (just as C_5 and C_6) vanish; this is assumed in the following.

For an interpretation of the results of solution of the dynamic problem and an explanation of the flow patterns in each of the fluids it is expedient to establish the dependence between the characteristic values of the variable $\varphi_{1u=0}, \varphi_{1\psi=0}, \varphi_{2u=0}, \varphi_{2\psi=0}$, corresponding to zero velocity of flow and zero stream function in the ranges $y > 0$ and $y < 0$, respectively, and the parameters m and Ω.

Setting equal to zero the lefthand sides of Eqs. (7.18) and (7.19) and carrying out some elementary transformations we obtain

$$\left.\begin{aligned} m &= \frac{\operatorname{erf}(\varphi_{1u=0}) + \Omega}{\operatorname{erf}(\varphi_{1u=0}) - 1}, \qquad m = \frac{\Omega\varphi_{1\psi=0} + \int\limits_0^{\varphi_{1\psi=0}}(\operatorname{erf} z)\,dz}{\int\limits_0^{\varphi_{1\psi=0}}(\operatorname{erf} z)\,dz - \varphi_{1\psi=0}}, \\ m &= \Omega\frac{\operatorname{erf}(\varphi_{2u=0}) + 1}{\Omega\operatorname{erf}(\varphi_{2u=0}) - 1}, \qquad m = \Omega\,\frac{\varphi_{2\psi=0} + \int\limits_0^{\varphi_{2\psi=0}}(\operatorname{erf} z)\,dz}{-\varphi_{2\psi=0} + \Omega\int\limits_0^{\varphi_{2\psi=0}}(\operatorname{erf} z)\,dz}. \end{aligned}\right\} \quad (7.20)$$

Figure 7.1 shows the velocity distributions calculated from Eq. (7.18) for several values of the parameters m and Ω. It follows rom the calculation that counterflows occur in the fluids at certain definite ratios of the parameters m and Ω in the ranges $y > 0$ and $y < 0$.

Note that in the limiting case $\Omega \to 0$ $(\mu_2 \to \infty)$ and $m = 0$ the first function of (7.18) will correspond to and approximate solution of the Blasius problem of a uniform laminar flow of incompressible fluid streaming around a plate.

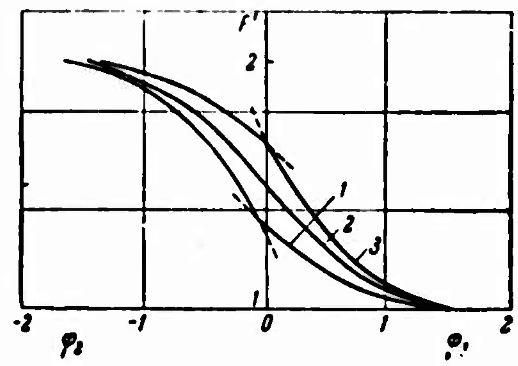

Fig. 7.1 Velocity distribution at the interface of immiscible fluids ($m = 2$). 1. $\Omega = 0,5$; 2. $\Omega = 1,0$; 3. $\Omega = 2,0$.

Let us now turn to the solution of the thermal problem. We introduce the new functions

$$\theta_1 = \theta_1(\varphi_1) = \frac{T^{(1)} - T_2}{T_1 - T_2}, \quad \theta_2 = \theta_2(\varphi_2) = \frac{T^{(2)} - T_2}{T_1 - T_2} \tag{7.21}$$

and rewrite the energy equations (7.8) and (7.9) and the boundary conditions:

$$u^{(1)} \frac{\partial \theta_1}{\partial x} + v^{(1)} \frac{\partial \theta_1}{\partial y} = a_1 \frac{\partial^2 \theta_1}{\partial y^2}, \quad u^{(2)} \frac{\partial \theta_2}{\partial x} + v^{(2)} \frac{\partial \theta_2}{\partial y} = a_2 \frac{\partial^2 \theta_2}{\partial y^2}, \tag{7.22}$$

$$\left.\begin{array}{l} \theta_1 = 1 \quad \text{if} \quad \varphi_1 = +\infty, \quad \theta_2 = 0 \quad \text{if} \quad \varphi_2 = -\infty, \\ \theta_1 = \theta_2, \; \lambda_1 \left.\dfrac{\partial \theta_1}{\partial y}\right|_{y=0} = \lambda_2 \left.\dfrac{\partial \theta_2}{\partial y}\right|_{y=0} \quad \text{if} \quad \varphi_1 = \varphi_2 = 0. \end{array}\right\} \tag{7.23}$$

Equation (7.22) for the determination of the functions θ_1 and θ_2 can be reduced to ordinary differential equations

$$\left.\begin{array}{l} \theta_1'' + 2Pr_1 F_1 \theta_1' = 0, \\ \theta_2'' + 2Pr_2 F_2 \theta_2' = 0, \end{array}\right\} \tag{7.24}$$

whose general solutions are easily obtained:

$$\theta_1(\varphi_1) = C_9 \int_0^{\varphi_1} (F_1'')^{Pr_1} d\varphi_1 + C_{11},$$

$$\theta_2(\varphi_2) = C_{10} \int_0^{\varphi_2} (F_2'')^{Pr_2} d\varphi_2 + C_{12}.$$

Determining the constants of integration C_9-C_{12} from the boundary conditions (7.23) we obtain finally

$$\left.\begin{aligned}\theta_1(\varphi_1) &= \frac{1}{1+\frac{\lambda_2}{\lambda_1}\sqrt{\frac{a_1}{a_2}}}\left[1+\frac{\lambda_2}{\lambda_1}\sqrt{\frac{a_1}{a_2}}\operatorname{erf}\left(\varphi_1\sqrt{Pr_1}\right)\right],\\ \theta_2(\varphi_2) &= \frac{1}{1+\frac{\lambda_2}{\lambda_1}\sqrt{\frac{a_1}{a_2}}}\left[1+\operatorname{erf}\left(\varphi_2\sqrt{Pr_2}\right)\right].\end{aligned}\right\} \tag{7.25}$$

Examples of the temperature distributions are shown in Fig. 7.2.

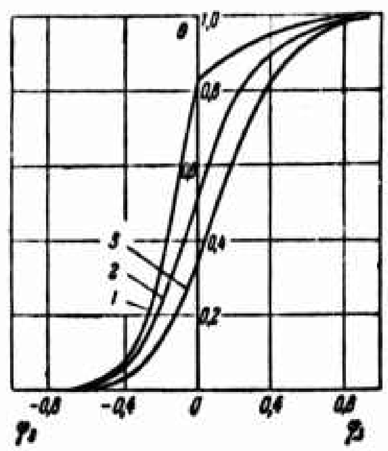

Fig. 7.2. Temperature distributions at the interface of immiscible fluids. ($Pr_1 = 4$, $Pr_2 = 9$). 1. $z = (\lambda_2/\lambda_1)\sqrt{a_1/a_2} = 2$; 2. $z = 1$; 3. $z = 0.2$.

To finish this section we give at last the expressions for the frictional stress and the heat flux at the interface of the fluids:

$$\tau = \sqrt{\frac{\rho_1\mu_1\rho_2\mu_2 u_1}{\pi x}}\,\frac{u_1-u_2}{\sqrt{\rho_1\mu_1}+\sqrt{\rho_2\mu_2}}, \tag{7.26}$$

$$q = -\frac{1}{2}\sqrt{\frac{u_1}{x}}\,(T_1-T_2)\,\frac{\lambda_1\lambda_2}{\lambda_1\sqrt{a_2}+\lambda_2\sqrt{a_1}}. \tag{7.27}$$

LITERATURE REFERENCES

25, 57, 114, 135, 156, 168, 174, 212, 287, 304.

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[Footnotes]

156 In the case of a twisted jet the self-similar solutions correspond — as this was shown previously — to the additional assumption of the so-called "slight twist."

Chapter 8

LAMINAR JETS OF A COMPRESSIBLE GAS

8.1 ON THE INFLUENCE OF COMPRESSIBILITY

The compressibility of a gas in jet flows, as also in other cases, may be the result of high values of the velocity of motion, a considerable heating and differences in the composition of the gas.

The influence of these three factors, velocity, temperature and composition of the gas, influence the jet flows of compressible gases both separately and jointly. The influence of all of them results in a change of density of the gas.

Restricting ourselves in this book to investigating unstressed flows we must, primarily, separate the following two problems: the influence of a variable density distribution (without reference to its origin) on the dynamic characteristics of the jet and the influence of the Prandtl number ν/a (or its analogue of the diffusion problem, sometimes denoted Schmidt's number ν/D) on the thermal (diffusion) characteristics.

In the framework of the theory of the expansion of laminar jets and the investigation by the methods of the boundary layer theory, it is quite natural that, first of all, one turns to solutions which permit a reduction of the problem of the motion of a jet of compressible gas to the equivalent problem of an incompressible fluid. These methods, which are connected with transformations of the flow from the physical plane to the plane of A.A. Dorodnitsyn's [92] or R. Mises' [212] var-

iables, are applied in the investigation of plane flows. For axisymmetric jets (disregarding fan-type jets) analogous methods of solution have not yet been developed. This fact forces us to restrict our considerations to problems of plane, free and semilimited jets and also fan-type jets.

The main problem which must be considered when solving problems on the expansion of jets of compressible gas results in a comparison of the laws of expansion of a gas jet issuing in a space filled by a gas of lower, the same or higher density. Speaking of the laws we mainly think of the variations of velocity, temperature and the like along the axis of the jet and in its cross sections, but also of variations of flow rate and other integral characteristics

In addition to this, it is, in particular for the thermal (diffusion) problem, necessary to elucidate the relationships between the velocity and temperature distributions, and also the effective thicknesses of the dynamic and thermal layers for various ratios of the coefficients of molecular momentum transfer (ν), heat transfer (a) and, generally, mass transfer (D). The quantities ν, a and D must be considered with regard to their dependences on the temperature (and the composition, when different gases mix). As to the pressure, it is in most cases possible to consider it to be constant.

When solving concrete problems, we restrict ourselves to the methods of the asymptotic layer applied to self-similar motions produced by source jets, and partially also to the method of small perturbations.

Before we discuss the methods of solving problems on the expansion of compressible gas jets, we devote ourselves to some general considerations of physical nature on the influence of compressibility on the expansion of jets.

For definiteness we consider the efflux from a nozzle of finite

dimensions of three gas jets (where the density ρ of the gas is lower, equal or higher than the density ρ_∞) of the same initial velocity into a space filled with a gas of the given density ρ_∞. It is quite natural to expect the damping of these jets, which may be characterized by, e.g., the decrease of axial velocity, to be the stronger the higher the ratio ρ_∞/ρ. In other words, a gas jet of low density (relative to the density of the medium) will be damped more rapidly than a gas jet whose density is the same as that of the surrounding medium. The axial velocity of a jet of a denser gas will decrease still more slowly. The same holds true for the mean flow rate values.

On the other hand, a gas jet of low density will expand more rapidly than in the case where the gas of the jet and the surrounding medium have equal densities. In contradistinction to this, a denser jet will expand only insignificantly. This is evident for the limiting case of the motion of a solid piston in a gaseous atmosphere.

Thus, the velocity (and also the temperature) along the axis and in the cross sections of a jet will decrease the more rapidly the higher the parameter of compressibility, $\omega = \rho_\infty/\rho$. This result is illustrated schematically in Fig. 8.1 and in greater detail in Fig. 8.2 (see below).

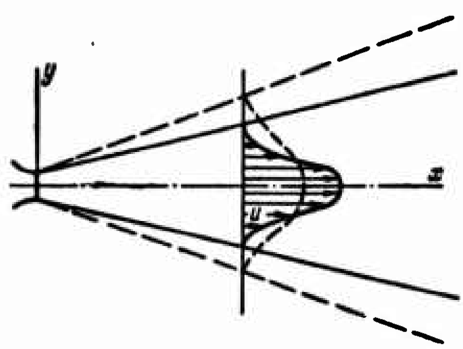

Fig. 8.1 Schematic representation of expansion of a compressible gas jet. —— Isothermal jet; – jet of hot gas.

Let us now derive in a general form the equations of a free laminar boundary layer of compressible gas, which will be used later on in order to solve severa concrete problems

The initial system of equations describing a steady, plane laminar flow in the absence of a pressure gradient can be written in the form

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right), \quad (8.1)$$

$$\frac{\partial p}{\partial y} = 0, \quad (8.2)$$

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0, \quad (8.3)$$

$$\rho u \frac{\partial i}{\partial x} + \rho v \frac{\partial i}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\lambda}{c_p}\frac{\partial i}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \quad (8.4)$$

(in the last equation the heat of friction has been taken into account).

We complete these equations by the equation of state

$$p = \frac{\varkappa - 1}{\varkappa}\rho i \quad (8.5)$$

and the equation linking the viscosity coefficient and the enthalpy

$$\mu = ci^n. \quad (8.6)$$

In these equations $i = c_p T$ is the enthalpy $c_p = \frac{\varkappa R}{\varkappa - 1}$, $\varkappa = \frac{c_p}{c_v}$, where c_p, c_v, $\varkappa$ and R are the specific heats with constant pressure and volume, their ratio and the specific gas constant respectively.

We also give another form of the energy equation which is often used; it takes the heat of friction into account. We introduce the enthalpy of damping

$$i_0 = i + \frac{u^2}{2}. \quad (8.7)$$

The energy equation for the quantity i_0 will then read

$$\rho u \frac{\partial i_0}{\partial x} + \rho v \frac{\partial i_0}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial i_0}{\partial y}\right) + \left(\frac{1}{Pr} - 1\right)\frac{\partial}{\partial y}\left(\mu \frac{\partial i}{\partial y}\right). \quad (8.8)$$

In the particular case of Pr = 1, Eq. (8.8) resembles Eq. (8.1). Under similar boundary conditions for velocity and enthalpy of damping Eq. (8.8) has the solution

$$i_0 = C_1 u + C_2. \quad (8.9)$$

In contrast to the system of equations for the incompressible fluid which permits an independent solution of the dynamic problem, the system of the equations of motion and energy for the compressible gas must be integrated as a whole.

As regards the case where a gas jet expands in a gaseous medium of different physical properties and where, consequently, molecular diffusion takes place, we restrict ourselves for the present (before considering a concrete problem) to the remark that the problem of the similarity of heat-conduction and diffusion problems is here more complex owing to the accounting for the heat of friction and the possibility that the values of the Prandtl numbers $Pr = \frac{\nu}{a}$, $Pr_{диф} = \frac{\nu}{D}$ and their ratio a/D = Le, the Lewis number.

It should also be noted that in the case of a motion with high velocity, owing to the commensurability of kinetic energy and heat content, the integral condition expressing the law of energy conservation must be written in terms of the total excessive enthalpy. For example, in the case of a plane jet of compressible gas we shall have

$$Q_0 = \int_{-\infty}^{+\infty} \rho u (i_0 - i_\infty) dy = \text{const.} \quad (8.10)$$

This equation is derived from the system of Eqs. (8.3) and (8.8) with the boundary condition for the plane jet

$$\frac{\partial i_0}{\partial y} = 0 \text{ if } y = \pm\infty.$$

In the case of a small velocity Eq. (8.10) will of course pass over to the previous condition of conservation of excessive heat content for an incompressible fluid.

The condition of conservation of the jet's momentum has the same form as in the case of an incompressible fluid:

$$J_x = \int_{-\infty}^{+\infty} \rho u^2 dy = \text{const.} \quad (8.11)$$

For other cases of flow (fan-type, semilimited and other gas jets) the corresponding integral invariants will be given together with the solutions of the concrete problems.

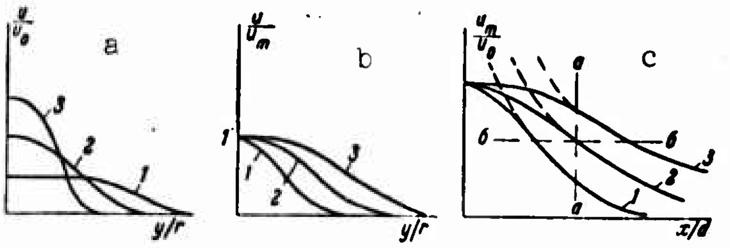

Fig. 8.2. Schematic representation of influence of the parameter of compressibility $\omega = (\rho_\infty/\rho)$ on the damping of a gas jet. a) Transverse velocity distributions with $x/d = \mathrm{idem}$; b) transverse velocity distributions with $u_m/u_0 = \mathrm{idem}$; c) variation of velocity along the jet axis $1. \omega > 1$; $2. \omega = 1$; $3. \omega < 1$.

Let us turn to the qualitative comparison of the laws of damping of a compressible gas jet with different values of the ratio of gas densities in the surrounding medium and the jet: $\omega = \frac{\rho_\infty}{\rho}$. Figure 8.2 shows the two variants of this comparison. In the first (Fig. 8.2a) the transverse velocity distributions are compared for one and the same cross section (for the same value of x/d, d being the nozzle parameter), for three jets with values of $\omega > 1$, $\omega = 1$ and $\omega < 1$ and the same value of the momentum flux J_x. In this case the ratio u_m/u_0 (with $y = 0$) will be smallest for the low density gas jet ($\omega > 1$) and highest for the gas jet whose density is high compared to that of the surrounding medium ($\omega < 1$). The effective boundary layer thicknesses will behave inversely: the thickness is greatest in the gas jet of lowest density and smallest in that of highest density.

In the second figure (Fig. 8.2b) the same velocity distributions are compared with the same value of u_m, i.e., for different distances (the corresponding values of x/d will be smallest for the low density jet and highest if $\omega < 1$). In this comparison the effective thickness

of the jet will be smallest in a jet of gas of low density ($\omega > 1$) and greatest in the case of $\omega < 1$. The choice of the curves shown in Fig . 8.2a and 8.2b is easy to trace with the help of the dependence of the axial velocity of the jet on the distance x/d shown in Fig. 8.2c. Figure 8.2a corresponds to the cross section "a-a" and Fig. 8.2b to cross section "b-b." A variation of the parameter ω exerts an analogous influence on the temperature distribution along the jet and in its cross sections (with the Prandtl number remaining unchanged).

From a qualitative point of view, we considered in detail the influence of the density ratio on the laws of velocity variation (and the variation of excessive temperature) as this problem is of great significance in principle and in practice. The qualitative image represented in Fig. 8.2 is essentially the same also for turbulent gas jets since it is based on a general and physically obvious situation: under otherwise equal conditions a jet of less dense gas is damped in a denser atmosphere more rapidly than in the opposite case.

8.2 ON THE ENERGY REDISTRIBUTION IN INTEGRAL-ADIABATIC FLOWS

Let us, from a qualitative point of view, consider the curious phenomenon of a local redistribution of the total enthalpy which arises in fast gas flows (in particular in laminar and turbulent jets of compressible gas) and also in the course of combustion in gases or on the surfaces they flow along, in the twisted gas flows of Rank-type tubes, etc. [47, 58]

In all cases of an essentially rectilinear* motion the nature of the effect, e.g., of a local redistribution of enthalpy and kinetic energy in adjacent gas jets, is mainly determined by the ratio of the kinetic coefficients characterizing the intensity of the processes of molecular (or molar) transfer of momentum (ν), heat (a) and substance

(D). In the case where two (or all three) effects coact in the general process, the ratio of these coefficients, the so-called Prandtl number, is the criterion for the local distribution of total enthalpy. In this case one of the processes of molecular exchange, the heat conduction, will in all cases be prominent in its role of a factor that tends to smooth the temperature distribution (that of the enthalpy $i = c_p T$). Two other processes, viscosity and diffusion, play the part of characteristics of heat liberation in the conversion of various forms of energy into heat content. In the case of a high velocity gas jet, we are concerned with the conversion of kinetic energy in enthalpy and dissipation of mechanical energy at the expense of the work of forces of internal friction. In the other particular case, the burning of gas, the latent chemical energy is converted into heat content. The general case where both factors of "heat liberation," viscosity and diffusion, coact and where, at the same time, heat is removed from the region in which the energy conversion takes place (the surface of a burning plate placed in the stream of a compressible gas) is considered in Paper [49].

In all cases, in particular in that of a jet flow of compressible gas, the physical nature of the effect will thus result in a peculiar "competition" of the local processes of energy conversion leading to the "appearance"of enthalpy ($c_p T$) on the one hand and to a leveling of its distribution by virtue of the thermal conductivity, on the other.

Here we restrict ourselves to the case of a high velocity flow of a compressible, chemically inert gas. The jet flow is assumed to develop from a gas flow uniform with respect to velocity and temperature, in a particular case, from the state of rest. The efflux is allowed to occur in a space filled by a quiescent gas of the same temperature. Such a flow may of course be called integral-adiabatic.

For greater illustrativeness we suppose conditionally that the

nonuniform velocity distribution in the jet is the result of a local adiabatic process. In this case it is obvious that the stagnation temperature T_0 should be the same in the entire field of flow. The thermodynamic temperature $T = T_0 - \frac{u^2}{2c_p}$ should of course be lowest in the gas jets of highest velocity and, *vice versa,* highest in the slowest jets. Let us now consider the kinetic interaction of the adjacent gas jets. Firstly, under the influence of a temperature gradient conductive heat flows will arise. Secondly, under the influence of internal friction on the sections of velocity variation heat will be set free.

In spite of the fact that the process as a whole takes place without heat exchange, which is determined by the boundary conditions, the constancy of the temperature T_0 at all boundaries of the flow region in the quiescent gas, the value of T_0 will only be the same in the whole field of flow if an additional condition is satisfied: the local equality ($\nu = a$) of the intensity of the processes of heat liberation and heat removal. If this equality is violated, e.g., in the case where the heat conduction is "faster" than the viscosity ($a > \nu$, Pr < 1), high velocity gas jets to which heat is supplied from hotter and slower flows will carry away the excessive total enthalpy (the sum of kinetic energy and heat content). Thus they behave as if they were enriched in t-tal enthalpy at the expense of the slower, impoverished flows. The opposite case, the enrichment of the slower and the impoverishment of the faster flows, is encountered with the inverse ratios ($\nu > a$, Pr > 1).

Thus, in the case of inequal transfer coefficient ν and a local extrema (maxima and minima) of total enthalpy will be observed in the field of flow. An example of this case is shown schematically in Fig. 8.3.

The above qualitative interpretation is generally valid and also applies to turbulent jets for which the "turbulent Prandtl number" is

smaller than unity.

The redistribution, shown schematically in Fig. 8.3, is verified by many calculations. Considered from the quantitative point of view,

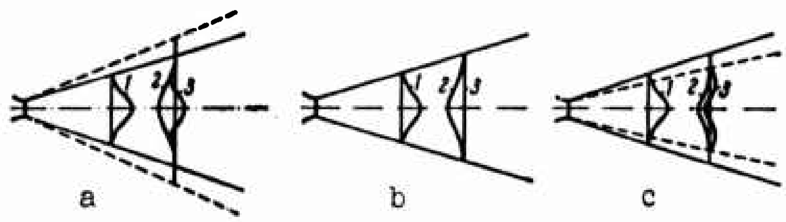

Fig. 8.3. Redistribution of total enthalpy in a high-velocity gas jet. a) Pr < 1; b) Pr = 1; c) Pr > 1; 1) velocity; 2) temperature; 3) stagnation temperature.

this effect, namely the deviation of the local value of the total enthalpy (or the stagnation temperature with $c_p = const$) proves relatively small. For example, for a laminar jet with the Mach number M = 1 the maximum deviation of the value of T_0 from $T_{0\infty}$ in the initial section of an integral-adiabatic jet amounts to about 2%. This effect will thus influence the dynamic problem only slightly (just as in general a difference in the Prandtl numbers) while its influence on the thermal problem, particularly for integral-adiabatic flows, is high. In the latter (measurement of temperature in fast gas streams, Rank effect, etc.) the nature of the effects is connected with the local redistributions of enthalpy.

Several examples of calculations of jets are given in the following.

8.3. THE PLANE SOURCE—JET. A.A. DORODNITSYN'S TRANSFORMATION

We use the equations of the plane laminar boundary layer of compressible gas given in the previous section in order to solve the problem of expansion of a free plane-parallel source jet in a space filled by the same gas.

The system of Eqs. (8.1)-(8.6) with the boundary conditions

$$\left.\begin{array}{ll}\frac{\partial u}{\partial y}=0, \quad v=0, \quad \frac{\partial i}{\partial y}=0 & \text{if } y=0, \\ u=0, \quad i=i_{\infty} & \text{if } y=\pm\infty,\end{array}\right\} \tag{8.12}$$

corresponding to the problem under consideration can hardly be integrated immediately. With the help of A.A. Dorodnitsyn's transformation of variables [92] the initial system of equations can be simplified considerably and reduced to a system of equations analogous to Eqs. (4.45) and (4.50) for the incompressible fluid. In the general case (with $p \neq$ const) these transformations have the form

$$\xi=\int_0^x \frac{p}{p_{\infty}} dx, \quad \eta=\int_0^y \frac{\rho}{\rho_{\infty}} dy$$

which, in the case of an isobaric jet flow pass over to the simpler transformations

$$\xi=x, \quad \eta=\int_0^y \frac{\rho}{\rho_{\infty}} dy. \tag{8.13}$$

The derivatives with respect to the old ("physical" variables x, y and A.A. Dorodnitsyn's variables ξ,η are linked by the formulas

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial \xi}+\frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial y}=\frac{\rho}{\rho_{\infty}}\frac{\partial}{\partial \eta}. \tag{8.14}$$

First we transform the continuity equation to the new variables. For this purpose, in the equations

$$\frac{\rho}{\rho_{\infty}} u=\frac{\partial \psi}{\partial y}, \quad \frac{\rho}{\rho_{\infty}} v=-\frac{\partial \psi}{\partial x},$$

which link the velocity components and the density with the stream-functions in the plane of the variables x,y, we use (8.14) to pass over to the variables ξ,η

$$u=\frac{\partial \psi}{\partial \eta}, \quad \tilde{v} \equiv \frac{\rho}{\rho_{\infty}} v+u\frac{\partial \eta}{\partial x}=-\frac{\partial \psi}{\partial \xi}. \tag{8.15}$$

Hence it follows that in the plane of A.A. Dorodnitsyn's variables ξ,η the continuity equation can be written in the form

$$\frac{\partial u}{\partial \xi}+\frac{\partial \tilde{v}}{\partial \eta}=0. \tag{8.16}$$

Transforming Eq. (8.1) we obtain

$$u\frac{\partial u}{\partial \xi}+\left(\frac{\rho}{\rho_\infty}v+u\frac{\partial \eta}{\partial x}\right)\frac{\partial u}{\partial \eta}=\frac{1}{\rho_\infty}\frac{\partial}{\partial \eta}\left(\frac{\mu\rho}{\rho_\infty}\frac{\partial u}{\partial \eta}\right). \quad (8.17)$$

Equations (8.5) and (8.6) are (with $p\neq$ const) equivalent to the following:

$$\rho i=\rho_\infty i_\infty, \quad \frac{\mu}{\mu_\infty}=\left(\frac{i}{i_\infty}\right)^n, \quad (8.18)$$

such that $\frac{\mu\rho}{\rho_\infty}=\mu_\infty\left(\frac{i}{i_\infty}\right)^{n-1}$ and Eq. (8.17) can be written in the form

$$u\frac{\partial u}{\partial \xi}+\tilde{v}\frac{\partial u}{\partial \eta}=\nu_\infty\frac{\partial}{\partial \eta}\left[\left(\frac{i}{i_\infty}\right)^{n-1}\frac{\partial u}{\partial \eta}\right]. \quad (8.19)$$

Equation (8.4) can be transformed in the following way:

$$u\frac{\partial i}{\partial \xi}+\tilde{v}\frac{\partial i}{\partial \eta}=\frac{\partial}{\partial \eta}\left(\frac{\lambda\rho}{\rho_\infty^2 c_p}\frac{\partial i}{\partial \eta}\right)+\nu_\infty\left(\frac{i}{i_\infty}\right)^{n-1}\left(\frac{\partial u}{\partial \eta}\right)^2.$$

But since $\frac{\lambda}{c_p}=\frac{\lambda}{\rho c_p}\frac{\mu}{\left(\frac{\mu}{\rho}\right)}=\frac{\mu}{Pr}$, we have $\frac{\lambda\rho}{\rho_\infty^2 c_p}=\frac{1}{Pr}\frac{\rho\mu}{\rho_\infty^2}=a_\infty\left(\frac{i}{i_\infty}\right)^{n-1}$ and the energy equation in the variables ξ,η takes the form

$$u\frac{\partial i}{\partial \xi}+\tilde{v}\frac{\partial i}{\partial \eta}=a_\infty\frac{\partial}{\partial \eta}\left[\left(\frac{i}{i_\infty}\right)^{n-1}\frac{\partial i}{\partial \eta}\right]+\nu_\infty\left(\frac{i}{i_\infty}\right)^{n-1}\left(\frac{\partial u}{\partial \eta}\right)^2. \quad (8.20)$$

The system of plane boundary layer equations (8.16), (8.19) and (8.20) obtained in the plane of A.A. Dorodnitsyn's variables remain so complicated that for a concrete problem, it may only be integrated numerically. As it is desirable to obtain an analytical solution to the problem of the plane source jet, we restrict ourselves to the simplest case of a linear dependence between the coefficient of viscosity μ and the enthalpy i. Thus we assume that $n = 1$. In this case the system of equations considered is simplified considerably:

$$\left.\begin{aligned}&u\frac{\partial u}{\partial \xi}+\tilde{v}\frac{\partial u}{\partial \eta}=\nu_\infty\frac{\partial^2 u}{\partial \eta^2},\\&\frac{\partial u}{\partial \xi}+\frac{\partial \tilde{v}}{\partial \eta}=0,\\&u\frac{\partial i}{\partial \xi}+\tilde{v}\frac{\partial i}{\partial \eta}=a_\infty\frac{\partial^2 i}{\partial \eta^2}+\nu_\infty\left(\frac{\partial u}{\partial \eta}\right)^2.\end{aligned}\right\} \quad (8.21)$$

A comparison of the first two equations of System (8.21) with the corresponding equations (4.45) of the dynamic problem for an incompressible fluid shows that they are identical as to their formulation. As regards the last equation of System (8.21) it differs from

the energy equation (4.50) only by the presence of the second term in the righthand side. Its origin is connected with the necessity of taking into account the influence of the heat of friction on the temperature distribution in fast gas flows. When a gas streams with a velocity considerably smaller than sonic velocity, this term will be small compared with the first term of the righthand side and can thus be neglected. In this case, the energy equations (4.50) and (8.21) also agree in their form.

Note that in the plane of A.A. Dorodnitsyn's variables the equations of the dynamic problem for a compressible gas can be integrated independently of the energy equation. The mutual relationship between the velocity and temperature distributions becomes apparent when we return to the physical plane of the variables x, y.

The boundary conditions for the problem on the plane source jet in new variables can be written in the form

$$\left.\begin{array}{ll} \frac{\partial u}{\partial \eta}=0, \quad \tilde{v}=0, \quad \frac{\partial i}{\partial \eta}=0 & \text{if } \eta=0, \\ u=0, \quad i=i_\infty & \text{if } \eta=\pm\infty. \end{array}\right\} \qquad (8.22)$$

The coincidence of the equations and boundary conditions of the dynamic problem mentioned above enables us to write immediately a solution to the system of the first two equations of (8.21) in the form

$$\left.\begin{array}{l} u(\xi,\eta)=A\xi^{\alpha}F'(\varphi_D)=\frac{1}{2}\sqrt[3]{\frac{3J_x^2}{4\rho_\infty^2\nu_\infty}}\,x^{-1/3}\,\frac{1}{\operatorname{ch}^2\varphi_D}, \\ \tilde{v}(\xi,\eta)=-\frac{A}{B}\xi^{\alpha-\beta-1}[(\alpha-\beta)F+\beta\varphi_D F']=\frac{1}{2}\sqrt[3]{\frac{\nu_\infty J_x}{6\rho_\infty x^2}}\,\frac{4\varphi_D-\operatorname{sh}2\varphi_D}{\operatorname{ch}^2\varphi_D}. \end{array}\right\} \qquad (8.23)$$

Here

$$\varphi_D=\frac{\eta}{2}\sqrt[3]{\frac{J_x}{6\rho_\infty\nu_\infty^2 x^2}}, \quad J_x=\int_{-\infty}^{+\infty}\rho u^2\,dy=\rho_\infty\int_{-\infty}^{+\infty}u^2\,d\eta.$$

When we neglect the heat of friction (which, as mentioned above, is admissible for flow velocities small compared to sonic velocity) we can write the energy equation in the variables ξ, η:

$$u\frac{\partial i}{\partial \xi}+\tilde{v}\frac{\partial i}{\partial \eta}=a_{\infty}\frac{\partial^2 i}{\partial \eta^2}, \qquad (8.24)$$

analogously to Eq. (4.50) for the incompressible fluid.

Equation (8.24) with the boundary conditions

$$\left.\begin{aligned}\frac{\partial i}{\partial \eta}&=0 \quad \text{if } \eta=0,\\ i&=i_{\infty} \quad \text{if } \eta=\pm\infty\end{aligned}\right\} \qquad (8.25a)$$

(symmetrical thermal boundary layer) or

$$\left.\begin{aligned}i&=i_1 \quad \text{if } \eta=+\infty,\\ i&=i_2 \quad \text{if } \eta=-\infty\end{aligned}\right\} \qquad (8.25b)$$

(asymmetrical thermal boundary layer) has, respectively, the solutions [see (4.56) and (4.60)]:

$$i=i_{\infty}+Q\sqrt[3]{\frac{2}{9\rho_{\infty}^2\nu_{\infty}J_x}}\left[\int_{-\infty}^{+\infty}(\operatorname{ch}\varphi_D)^{-2(Pr+1)}\,d\varphi_D\right]^{-1}x^{-1/3}(\operatorname{ch}\varphi_D)^{-2Pr}, \qquad (8.26a)$$

$$i=i_2+(i_1-i_2)\int_{-\infty}^{\varphi_D}(\operatorname{ch}\varphi_D)^{-2Pr}\,d\varphi_D\left[\int_{-\infty}^{+\infty}(\operatorname{ch}\varphi_D)^{-2Pr}\,d\varphi_D\right]^{-1}. \qquad (8.26b)$$

The symbol Q in Eq.(8.26a) denotes the flux of excess heat content in the jet,

$$Q=\int_{-\infty}^{+\infty}\rho u(i-i_{\infty})\,dy=\rho_{\infty}\int_{-\infty}^{+\infty}u(i-i_{\infty})\,d\eta=\text{const.} \qquad (8.27)$$

In order to analyze the results of solution, it is necessary to return from A.A. Dorodnitsyn's variables ξ,η to the physical plane of variables x,y. This transition is carried out with the help of Eq. (8.13) and also (8.26a) or (8.26b). In fact, it follows from (8.13) that

$$x=\xi, \quad y=\int_0^{\eta}\frac{\rho_{\infty}}{\rho}\,d\eta=\int_0^{\eta}\frac{i}{i_{\infty}}\,d\eta. \qquad (8.28)$$

Consequently, in the case of the boundary conditions (8.25a) the transition formulas will read

$$x=\xi, \quad y=\eta+\frac{\Gamma}{i_{\infty}}x^{-1/3}\int_0^{\eta}\frac{d\eta}{(\operatorname{ch}\varphi_D)^{2Pr}} \qquad (8.29)$$

or

$$x = \xi, \quad \varphi = \varphi_D + \frac{\Gamma}{i_\infty} x^{-1/3} \int_0^{\varphi_D} \frac{d\varphi_D}{(\operatorname{ch} \varphi_D)^{2Pr}}$$

$$\left(\text{here } \Gamma = Q \sqrt[3]{\frac{2}{9\rho_\infty^2 \nu_\infty J_x}} \left[\int_{-\infty}^{+\infty} (\operatorname{ch} \varphi_D)^{-2(Pr+1)} d\varphi_D \right]^{-1}\right). \tag{8.30}$$

In the case of an asymmetrical thermal boundary layer, we obtain the transformation formulas from expressions analogous to (8.28):

$$x = \xi, \quad y = \int_0^\eta \frac{i}{i_1} d\eta \tag{8.31}$$

or

$$x = \xi, \quad \varphi = \int_0^{\varphi_D} \frac{i}{i_1} d\varphi_D. \tag{8.32}$$

Using Eq. (8.26b) we find the sought relation between the independent variables:

$$x = \xi, \; \varphi = \frac{i_2}{i_1} \varphi_D + \left(1 - \frac{i_2}{i_1}\right) \left[\int_{-\infty}^{+\infty} (\operatorname{ch} \varphi_D)^{-2Pr} \right]^{-1} \int_0^{\varphi_D} \left(\int_{-\infty}^{z} (\operatorname{ch} z)^{-2Pr} dz \right) dz. \tag{8.33}$$

Formulas (8.23), (8.26a), (8.26b) together with (8.30) and (8.33) yield a solution to the problem in the physical plane of the variables x, y; knowing the succession of the values of the variable φ_D we also know the succession of the values of the variable φ and also $u(\xi, \varphi_D)$ and $i(\xi, \varphi_D)$.

An analysis of the results of solution shows that, as in other cases of jet flows, a change of the values of Prandtl's number has hardly an influence on the velocity distribution but alters the temperature distribution essentially. An increase of the Prandtl number Pr causes a slight broadening of the dynamic boundary layer and a strong reduction in thickness of the thermal boundary layer.

With asymmetric conditions for the thermal boundary layer a variation of the parameter i_2/i_1 results in a distortion of the velocity distribution which is symmetrical with respect to the axis Ox (the lines

$y = 0$ or $\varphi = 0$) only if $i_2/i_1 = 1$. If $i_1/i_2 < 1$ the velocity distribution becomes narrower than in the isothermal jet ($i_1 = i_2$), if $i_1/i_2 > 1$ it is broader.

On the basis of the solutions obtained, it is easy to determine the laws governing the variation of the jet's integral characteristics. In particular, the mass flow per second in a jet with symmetrical or asymmetrical boundary conditions for the enthalpy is given by the formula

$$G_m = \int_{-\infty}^{+\infty} \rho u \, dy = \rho_\infty \int_{-\infty}^{+\infty} u \, d\eta = \sqrt[3]{36 \rho_\infty^2 \nu_\infty J_x x},$$

which is identical with the analogous relation for the incompressible gas.

The volume flow rate for a symmetric thermal layer is equal to

$$G_v^{\text{сим}} = \int_{-\infty}^{+\infty} u \, dy = \int_{-\infty}^{+\infty} \frac{i}{i_\infty} u \, d\eta = \sqrt[3]{36 \frac{\nu_\infty J_x}{\rho_\infty}} \left[x^{1/3} + \frac{\Gamma}{2i_\infty} \int_{-\infty}^{+\infty} (F')^{Pr+1} \, d\varphi_D \right].$$

In the case of an asymmetric thermal boundary layer

$$G_v^{\text{асим}} = \int_{-\infty}^{+\infty} \frac{i}{i_1} u \, d\eta = \frac{i_2}{i_1} \sqrt[3]{36 \frac{\nu_\infty J_x}{\rho_\infty} x} \left\{ 1 + \left(\frac{i_1}{i_2} - 1 \right) \times \right.$$

$$\left. \times \left[\int_{-\infty}^{+\infty} F'^{Pr} d\varphi_D \right]^{-1} \int_0^{\varphi_D} F' \left(\int_{-\infty}^{\varphi_D} F'^{Pr} \, d\varphi_D \right) d\varphi_D \right\}.$$

For a jet of incompressible gas $\left(\Gamma \to 0, \frac{i_2}{i_1} \to 1\right)$ these two expressions yield

$$G_{v\,\text{несж}} = \sqrt[3]{36 \frac{\nu_\infty J_x}{\rho_\infty} x}.$$

Note by the way that at great distances from the source and in a jet of compressible gas $G_v^{\text{сим}} \to G_{v,\,\text{несж}}$ (since the second term in the brackets is constant and the first increases with x); in contrast to this $G_v^{\text{асим}}$ displays no such tendency as both terms in the righthand side vary likewise as x is increased.

The local Nusselt number

$$\mathrm{Nu}_x = \frac{1}{2\sqrt{6}}\left[\int_{-\infty}^{+\infty} (\mathrm{ch}\,\varphi_D)^{-2\mathrm{Pr}}\, d\varphi_D\right]^{-1} (\mathrm{Re}_x)^{1/2} \quad \left(\mathrm{Re}_x = \frac{xJ_x}{\rho_\infty \nu_\infty^2}\right)$$

is determined by the same formula as in the case of an incompressible gas (see Section 4.3).

An analogous method can be used to solve the problem of the fan-type, slightly twisted jet of compressible gas with the two variants (symmetric and asymmetric) of boundary conditions for the temperature [32]. Without giving the solution we only remark that in the plane of A.A. Dorodnitsyn's variables, it agrees with the solution of the corresponding problem for an incompressible fluid. As to the peripheral velocity component, its distribution in such a jet will resemble the distribution of the radial velocity component.

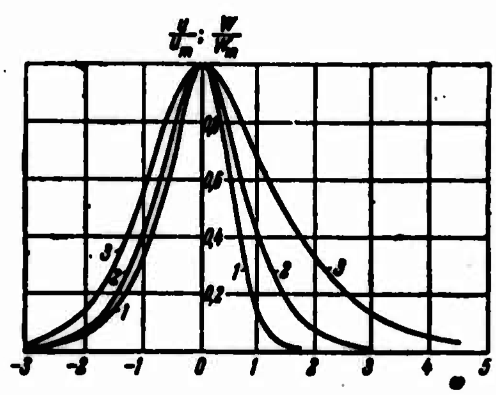

Fig. 8.4. Transverse distribution of the longitudinal and peripheral velocity components in a fan-type jet of gas with asymmetric boundary conditions for the temperature. *1.* $(i_1/i_2) = 1/2$; *2.* $(i_1/i_2) = 1$; *3.* $(i_1/i_2) = 2$.

An essential difference in the behavior of jets of incompressible fluid and compressible gas become apparent when we return to the plane of physical variables. In particular, the temperature (density) distribution exerts an inverse influence on the velocity distributions. This becomes particularly distinct if we consider the example of the fan-type or plane gas jet with asymmetric boundary conditions for the temperature:

a velocity distribution which is symmetrical in A.A. Dorodnitsyn's variables, in a jet cross section, becomes asymmetric in the plane of x, y, and so on.

To illustrate these conditions we give some examples of calculation of velocity and temperature distributions in cross sections of a twisted fan-type jet of gas of low velocity in Figs. 8.4-8.6.

8.4 SOLUTION TAKING THE HEAT OF FRICTION INTO ACCOUNT

Let us now consider the solution of the problem of a plane source jet where the heat of friction is taken into account. In this case in the initial system of Eqs. (8.21), written in A.A. Dorodnitsyn's variables, the energy equation must be replaced by the following equation:

$$u\frac{\partial i_0}{\partial \xi}+\tilde{v}\frac{\partial i_0}{\partial \eta}=\nu_\infty\frac{\partial^2 i_0}{\partial \eta^2}+\left(\frac{1}{Pr}-1\right)\nu_\infty\frac{\partial^2 i}{\partial \eta^2} \quad (8.34)$$

($i_0=i+\frac{u^2}{2}$ is the stagnation enthalpy).

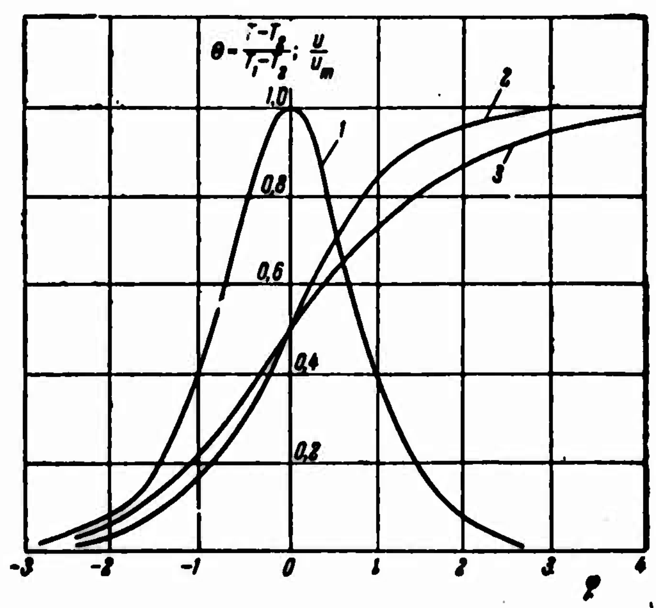

Fig. 8.5 Transverse distribution of excess temperature in fan-type gas jet. 1) Velocity ($i_1 = i_2$); 2) temperature ($i_1 = i_2$; $\rho = const$); 3) temperature ($i_1 = 2\ i_2$).

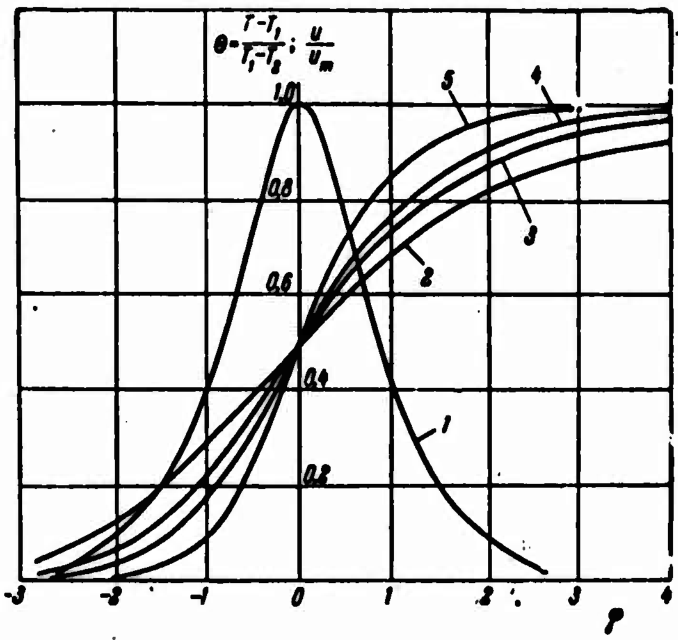

Fig. 8.6. Transverse temperature distribution in fan-type gas jet. 1) Velocity; 2-5) Temperature; *2. Pr = 0.5; 3. Pr = 0.75; 4. Pr = 1.0; 5. Pr = 2.0.*

To simplify the solution of the problem we assume that Pr = 1. This limitation proves unessential for the following two reasons. Firstly, the Prandtl number for air (and a series of other gases) is sufficiently close to unity. Secondly, the influence of a change of the Prandtl number on the velocity and temperature distributions has already been considered above for the case of a jet of low velocity; this influence remains the same in qualitative respects also in the case of a jet of high velocity.

With Pr = 1, Eq. (8.34) can be rewritten in the form

$$u\frac{\partial i_0}{\partial \xi} + \tilde{v}\frac{\partial i_0}{\partial \eta} = \nu_\infty \frac{\partial^2 i_0}{\partial \eta^2}. \qquad (8.35)$$

Comparing the first equation of System (8.21) with Eq. (8.35) results, as already mentioned above, in the conclusion that i_0 is a linear function of u:

$$i_0 = C_1 u + C_2. \qquad (8.36)$$

The constant C_2 is determined from the boundary conditions which we write in the form

$$\left.\begin{aligned} &\frac{\partial i_0}{\partial \eta}=0 \quad \left(\frac{\partial i}{\partial \eta}=0\right) \text{ if } \eta=0, \\ &i_0=i_\infty \qquad\qquad\qquad \text{if } \eta=\pm\infty. \end{aligned}\right\} \tag{8.37}$$

Hence it follows that $C_2 = i_\infty$. Since the boundary conditions (8.37) are insufficient for the determination of the constant C_1 we must use an integral invariant which is easy to derive. For this purpose, we rewrite Eq. (8.35)

$$u\frac{\partial}{\partial \xi}(i_0-i_\infty)+\tilde{v}\frac{\partial}{\partial \eta}(i_0-i_\infty)=\nu_\infty\frac{\partial^2}{\partial \eta^2}(i_0-i_\infty)$$

and with the help of the continuity equation $\frac{\partial u}{\partial \xi}+\frac{\partial \tilde{v}}{\partial \eta}=0$ it can be given the form

$$\frac{\partial}{\partial \xi}[u(i_0-i_\infty)]+\frac{\partial}{\partial \eta}[\tilde{v}(i_0-i_\infty)]=\nu_\infty\frac{\partial^2}{\partial \eta^2}(i_0-i_\infty).$$

Integrating it with respect to η from $-\infty$ to $+\infty$, taking the boundary conditions (8.37) into account, we arrive at

$$\frac{d}{d\xi}\int_{-\infty}^{+\infty}u(i_0-i_\infty)\,d\eta=0$$

or

$$\rho_\infty\int_{-\infty}^{+\infty}u(i_0-i_\infty)\,d\eta=Q_0=\text{const}. \tag{8.38}$$

Thus, together with the momentum of the jet, the value of the flux of the excess total enthalpy is also conserved.

Substituting the value of i_0 from (8.36) in Condition (8.38)

$$C_1\rho_\infty\int_{-\infty}^{+\infty}u^2\,d\eta=Q_0;$$

and taking into account that $\rho_\infty\int_{-\infty}^{+\infty}u^2\,d\eta=J_x,$ we obtain

$$C_1=\frac{Q_0}{J_x}$$

and, consequently,

$$i=i_\infty-\frac{u^2}{2}+\frac{Q_0}{J_x}u. \tag{8.39}$$

Since the dynamic equations and the boundary conditions for the velocity in the case considered are maintained in precisely the form of (8.21) and (8.22), the solution will also have the previous form (8.23).

The expressions for the velocity components (8.23) and Enthalpy (8.39) constitute a solution to the problem of the "fast" jet of compressible gas in A.A. Dorodnitsyn's variables. To complete the solution it is necessary to perform the inverse transition, i.e., to return to the variables x, y. This may be achieved with the help of formulas analogous to (8.28):

$$x = \xi, \quad \varphi = \int_0^{\varphi_D} \frac{i}{i_\infty} d\varphi_D,$$

and also Expressions (8.39)

$$\varphi = \varphi_D - \frac{1}{2i_\infty} \int_0^{\varphi_D} u^2 d\varphi_D + \frac{Q_0}{J_x i_\infty} \int_0^{\varphi_D} u \, d\varphi_D. \tag{8.40}$$

We calculate each of the integrals in the righthand side of the latter equation, using the expression for the longitudinal velocity component of (8.23):

$$\int_0^{\varphi_D} u^2 d\varphi_D = A^2 \xi^{-2/3} \int_0^{\varphi_D} \frac{dz}{\mathrm{ch}^4 z} = \frac{1}{3} A^2 \xi^{-2/3} (2 + \mathrm{ch}^{-2} \varphi_D) \, \mathrm{th} \, \varphi_D,$$

$$\int_0^{\varphi_D} u \, d\varphi_D = A \xi^{-1/3} F(\varphi_D) = A \xi^{-1/3} \, \mathrm{th} \, \varphi_D.$$

Consequently,

$$x = \xi,$$

$$\varphi = \varphi_D - \frac{J_x}{48 \rho_\infty i_\infty} \sqrt[3]{\frac{9 J_x}{2 \rho_\infty \nu_\infty^2}} \, \xi^{-2/3} (2 + \mathrm{ch}^{-2} \varphi_D) \, \mathrm{th} \, \varphi_D + \tag{8.41}$$

$$+ \frac{Q_0}{2 \rho_\infty i_\infty} \sqrt[3]{\frac{3 \rho_\infty}{4 \nu_\infty J_x}} \, \xi^{-1/3} \, \mathrm{th} \, \varphi_D.$$

Formulas (8.23), (8.39) and (8.41) enable us to construct the velocity and temperature distribution in the physical coordinates x, y. Figure 8.7, taken from a paper by Yu.T. Reznichenko [160] shows

the velocity distributions in functions of the nondimensional parameter $\overline{M}=\frac{Q_0}{J_x\sqrt{i_\infty}}$ which characterizes the heating of the jet. The higher the temperature difference between jet and surrounding medium $T-T_\infty=\frac{1}{C_p}(i-i_\infty)$, the higher, under otherwise equal conditions, are the quantities Q_0 and $\overline{M}$ and the higher, as may be seen from Fig. 8.7, are the effective thicknesses of the thermal and dynamic boundary layers.

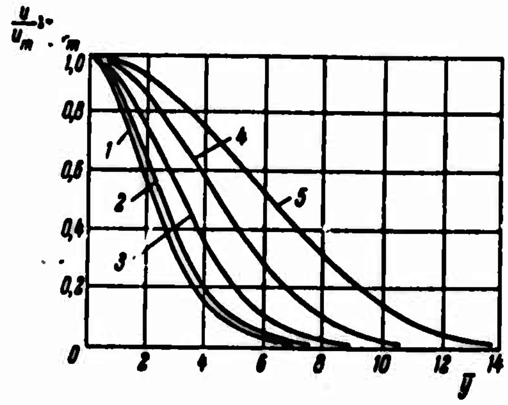

Fig. 8.7. Velocity distributions in a gas [160] jet where the heat of friction has been taken into account; Pr = 1). 1. $\overline{M}=0$; 2. $\overline{M}=0{,}5$; 3. $\overline{M}=1{,}0$; 4. $\overline{M}=1{,}5$; 5. $\overline{M}=2{,}0$.

Let us determine the expressions for the other integral characteristics of a compressible gas jet.

As already mentioned, the mass flow in a jet of compressible gas is determined by the same expression as in the case of an incompressible gas. In other words, with the same momenta and characteristics of the surrounding quiescent gas the mass flows in jets of compressible gas and such of imcompressible gas will be the same.

The volume mass flow will be equal to

$$G_v=\int_{-\infty}^{+\infty}u\,dy=\rho_\infty\int_{-\infty}^{+\infty}\frac{u}{\rho}\,d\eta=\int_{-\infty}^{+\infty}\frac{i}{i_\infty}u\,d\eta, \qquad (8.42)$$

but as from (8.39) we find $\frac{i}{i_\infty}=1-\frac{u^2}{2i_\infty}+\frac{Q_0}{J_x i_\infty}u$, we have consequently

$$G_v=\int_{-\infty}^{+\infty}u\,d\eta-\frac{1}{2i_\infty}\int_{-\infty}^{+\infty}u^3\,d\eta+\frac{Q_0}{J_x i_\infty}\int_{-\infty}^{+\infty}u^2\,dy.$$

Calculating the integral, we obtain

$$\int_{-\infty}^{+\infty} u\,d\eta = \sqrt[3]{\frac{36\nu_\infty J_x x}{\rho_\infty}},$$

$$\int_{-\infty}^{+\infty} u^2\,d\eta = \frac{J_x}{\rho_\infty},$$

$$\int_{-\infty}^{+\infty} u^3\,d\eta = \frac{1}{5}\sqrt[3]{\frac{6J_x^5}{\rho^5 \nu_\infty x}}.$$

We therefore arrive at the following expression for the volume flux in a jet of compressible gas

$$G_v = \sqrt[3]{36\frac{\nu_\infty J_x}{\rho_\infty}x}\left\{1 - \frac{J_x}{10\rho_\infty i_\infty}\sqrt[3]{\frac{J_x}{6\rho_\infty \nu_\infty^2 x^3}}\right\} + \frac{Q_0}{\rho_\infty i_\infty} \qquad (8.43)$$

which differs from the volume flux in a jet of incompressible gas. At a great distance from the nozzle the second term in the braces may become negligibly small compared to unity such that

$$G_{v,\text{сж}} \cong G_{v,\text{несж}} + \frac{Q_0}{\rho_\infty i_\infty}. \qquad (8.44)$$

Expression (8.43) also enables us to estimate the influence of the various parameters on the thickness of the jet. The fact is that an increase of the parameters J_x and i_∞ with fixed values of the other quantities results in a decrease of the volume flux and, consequently, in a decrease of the effective thickness of the jet. An increase of the quantity Q_0, however, has the inverse effect. These conclusions agree with those drawn above.

8.5. SOLUTION IN R. MISES' VARIABLES

Let us illustrate another method widespread in the theory of compressible gas jets by way of example of the solution obltained by D. Toose [318] for a laminar plane source jet of compressible gas. We are here concerned with a transformation of variables suggested by R. Mises [212].

In order to integrate the system of equations of a plane boundary

layer of compressible gas (with Pr = 1)

$$\left.\begin{aligned}&\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right),\\&\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0,\\&\rho u\frac{\partial\Delta i_0}{\partial x}+\rho v\frac{\partial\Delta i_0}{\partial y}=\frac{\partial}{\partial y}\left(\mu\frac{\partial\Delta i_0}{\partial y}\right),\\&\rho i=\rho_\infty i_\infty,\quad\frac{\mu}{\mu_\infty}=\frac{i}{i_\infty}\quad(\Delta i_0=i_0-i_\infty)\end{aligned}\right\}\tag{8.45}$$

we introduce the new variables ξ,ψ instead of x,y, where $\xi = x$ and ψ is the stream function defined by the equations

$$\frac{\rho}{\rho_\infty}u=\frac{\partial\psi}{\partial y},\quad\frac{\rho}{\rho_\infty}v=-\frac{\partial\psi}{\partial x}.\tag{8.46}$$

The derivatives transform according to the formulas

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial\xi}-\frac{\rho}{\rho_\infty}v\frac{\partial}{\partial\psi},\quad\frac{\partial}{\partial y}=\frac{\rho}{\rho_\infty}u\frac{\partial}{\partial\psi}.$$

Passing over to the new variables, we rewrite the first equation of System (8.45) in the form

$$\frac{\partial u}{\partial\xi}=\nu_\infty\frac{\partial}{\partial\psi}\left(u\frac{\partial u}{\partial\psi}\right).\tag{8.47}$$

With similar boundary conditions for velocity and total enthalpy and Pr = 1 the energy equation has (as above) the solution

$$\Delta i_0=C_1u+C_2.\tag{8.48}$$

The problem has thus been reduced to an integration of Eq. (8.47) and the determination of the constants of integration (among them also Eq. (8.48) with the boundary conditions

$$\left.\begin{aligned}&\frac{\partial u}{\partial\psi}=0,\quad\frac{\partial\Delta i_0}{\partial\psi}=0&&\text{if }\psi=0,\\&u=0,\quad\Delta i_0=0,\quad\frac{\partial u}{\partial\psi}=0&&\text{if }\psi=\pm\psi_1.\end{aligned}\right\}\tag{8.49}$$

We represent the longitudinal velocity component in the form

$$u=Ax^\alpha F(\varphi),\quad\varphi=B\psi\xi^\beta;\tag{8.50}$$

and, putting

$$\beta=-\frac{\alpha+1}{2},\quad AB^2=\frac{1}{6\nu_\infty},\tag{8.51}$$

Equation (8.47) can be transformed to the ordinary differential equation

$$(FF')' + 3[(\alpha+1)\varphi F' - 2\alpha F] = 0. \qquad (8.52)$$

A determination of the self-similarity constant α and the constants A and B is only possible with the help of an integral invariant which is easy to obtain from (8.47):

$$\rho_\infty \int_{-\psi_1}^{+\psi_1} u\, d\psi_i^* = R_u = \text{const}. \qquad (8.53)$$

From (8.53), taking (8.50) into account, we obtain

$$\alpha = \beta, \quad \frac{A}{B} = \frac{R_u}{\rho_\infty J} \quad \left(J = \int_{-\varphi_1}^{+\varphi_1} F d\varphi\right). \qquad (8.54)$$

The numerical values of the limits of integration of φ_1 will be determined in the following after we have found the form of the function $F(\varphi)$.

Consequently, we obtain from (8.51) and (8.54)

$$\alpha = \beta = -\frac{1}{3}, \quad A = \sqrt[3]{\frac{R_u^2}{6\rho_\infty^2 \nu_\infty J^2}}, \quad B = \sqrt[3]{\frac{\rho_\infty J}{6\nu_\infty R_u}}. \qquad (8.55)$$

Thus, in order to obtain the function $F(\varphi)$ we must integrate the very simple equation

$$(FF')' + 2(\varphi F)' = 0 \qquad (8.55a)$$

with the boundary conditions

$$\begin{aligned} F &= 1 \text{ if } \quad \varphi = 0, \\ F &= 0 \text{ if } \quad \varphi = \pm 1. \end{aligned} \qquad (8.56)$$

The solution of Eq. (8.55a) has the form

$$F = 1 - \varphi^2. \qquad (8.57)$$

Since at the boundaries of the boundary layer (that is, at $y = \pm\infty$) the longitudinal velocity component u must vanish, it follows from Eqs. (8.50) and (8.57) that the variable φ may only vary within the limits $-1 < \varphi < +1$. This yields $J = \int_{-1}^{+1} (1-\varphi^2)\, d\varphi = \frac{4}{3}$.

Let us turn to the determination of the constants C_1 and C_2 entering the expression for the total enthalpy (8.48).

It follows from the boundary conditions (8.49) that in Eq. (8.48) the constant $C_2 = 0$. The second constant $C_1 \neq 0$ is determined by an integral condition which is obtained when in the energy equation (8.45) we pass over to the variables ξ, ψ given by R. Mises:

$$\frac{\partial}{\partial \xi}(\Delta i_0) = \nu_\infty \frac{\partial}{\partial \psi}\left(u \frac{\partial \Delta i_0}{\partial \psi}\right). \qquad (8.58)$$

The integration of the latter equation yields

$$\rho_\infty \int_{-\psi_1}^{+\psi_1} \Delta i_0 \, d\psi = R_i = \text{const}; \qquad (8.59)$$

from which, when taking (8.48) into account, we find

$$C_1 = \frac{R_i}{R_u}$$

and

$$\Delta i_0 = \frac{R_i}{R_u} u. \qquad (8.60)$$

We finally obtain in the variables ξ, ψ

$$\left.\begin{aligned} u &= \frac{1}{2}\sqrt[3]{\frac{3R_u^2}{4\rho_\infty^2 \nu_\infty}}\, x^{-1/3}(1-\varphi^2),\\ i_0 &= i_\infty + \frac{R_i}{R_u} u = i_\infty + \frac{R_i}{2\sqrt[3]{\frac{4}{3}\rho_\infty^2 \nu_\infty R_u}}\, x^{-1/3}(1-\varphi^2),\\ \varphi &= \psi \sqrt[3]{\frac{2\rho_\infty}{9\nu_\infty R_u x}}, \quad \xi = x. \end{aligned}\right\} \qquad (8.61)$$

In order to transform Solution (8.61) to the physical variables x, y we use the first equation of (8.46). With the value of the coordinate x fixed we obtain

$$y = \int_0^\psi \frac{\rho_\infty}{\rho} \frac{d\psi}{u} = \frac{\xi^{1/3}}{B} \int_0^\varphi \frac{i}{i_\infty} \frac{d\varphi}{u}. \qquad (8.62)$$

From the second equation of (8.61) it follows that

$$\frac{i}{i_\infty} = 1 + \frac{R_i}{R_u i_\infty} u - \frac{u^2}{2 i_\infty}.$$

We therefore write Eq. (8.62) in the form

$$y = \frac{1}{B}\xi^{1/3}\left\{\int_0^\varphi \frac{d\varphi}{u} + \frac{R_i}{R_u i_\infty}\varphi - \frac{1}{2 i_\infty}\int_0^\varphi u \, d\varphi\right\}. \qquad (8.63)$$

Calculating the integral, we obtain

$$y=\frac{1}{B}\xi^{1/3}\left\{\frac{1}{2A}\xi^{1/3}\ln\frac{1+\varphi}{1-\varphi}+\frac{R_1}{R_u i_\infty}\varphi-\frac{A}{2i_\infty}\xi^{-1/3}\varphi\left(1-\frac{1}{3}\varphi^2\right)\right\}$$

and, finally,

$$y=\sqrt[3]{6\frac{\rho_\infty v_\infty^2}{R_u}}\,x^{2/3}\ln\frac{1+\varphi}{1-\varphi}+\frac{R_1}{i_\infty}\sqrt[3]{\frac{9v_\infty}{2\rho_\infty R_u^2}}\,x^{1/3}\varphi-\frac{3R_u}{8\rho_\infty i_\infty}\varphi\left(1-\frac{1}{3}\varphi^2\right). \quad (8.64)$$

From the latter relation it follows that $-1<\varphi<+1$ with $-\infty<y<+\infty$.

Together with (8.64) Expression (8.61) yields a solution of the problem on the plane source jet of compressible gas. The corresponding velocity and temperature distributions are given in *D* [318].

At the end of this section, we want to establish the law of increase of the boundary layer thickness in the source jet. As the quantity that characterizes the half-width of the jet at a given distance from the source, we choose this distance *b* from the jet axis at which the velocity is equal to half the maximum value of the velocity in the cross section consi-ered. From the first equation of (8.61) it follows that

$$u=\frac{1}{2}u_m \text{ with } \varphi=\pm\sqrt{\frac{1}{2}};$$

and then we obtain from (8.64) the following law of variation of the half-width of the jet:

$$b(x)=\sqrt[3]{\frac{6\rho_\infty v_\infty^2}{R_u}}\,x^{2/3}\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}+\frac{R_1}{\sqrt{2}i_\infty}\sqrt[3]{\frac{9v_\infty}{2\rho_\infty R_u^2}}\,x^{1/3}-\frac{15\sqrt{2}}{48}\frac{R_u}{\rho_\infty i_\infty}.$$

8.6. THE MIXING OF UNIFORM FLOWS OF COMPRESSIBLE GAS

The problem of the mixing of parallel and antiparallel flows of incompressible fluids dealt with in Section 5.3 permits a generalization to the case of the motion of compressible gas flows [118]. The solution of this more general problem is considered in the following.

The system of equations of a plane laminar boundary layer of com-

pressible gas

$$\left.\begin{aligned}
&\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right),\\
&\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0,\\
&\rho u \frac{\partial i}{\partial x} + \rho v \frac{\partial i}{\partial y} = \frac{1}{Pr}\frac{\partial}{\partial y}\left(\mu \frac{\partial i}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2,\\
&p = \frac{\varkappa - 1}{\varkappa}\rho i, \quad \mu = ci^n,
\end{aligned}\right\} \tag{8.65}$$

must in our case be integrated with the following boundary conditions:

$$\left.\begin{aligned}
u = u_1, \quad i = i_1 \text{ with } y = +\infty,\\
u = u_2, \quad i = i_2 \text{ with } y = -\infty.
\end{aligned}\right\} \tag{8.66}$$

We transform Eq. (8.65) and the boundary conditions (8.66) to a nondimensional form, using as scale units the quantities connected with the nonperturbed flow moving in the range of $y > 0$:

$$\bar{u} = \frac{u}{u_1}, \quad \bar{v} = \frac{v}{u_1}\sqrt{\mathrm{Re}}, \quad \bar{i} = \frac{i}{i_1}, \quad \bar{\rho} = \frac{\rho}{\rho_1}, \quad \bar{\mu} = \frac{\mu}{\mu_1},$$
$$\bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{L}\sqrt{\mathrm{Re}} \quad \left(\mathrm{Re} = \frac{\rho_1 u_1 L}{\mu_1}\right)$$

(L – scale unit of length).

When the pressure is assumed to be constant in the entire field of flow, the equations can be written in nondimensional variables in the form (we omit the bars on the nondimensional quantities):

$$\left.\begin{aligned}
&\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right),\\
&\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0,\\
&\rho u \frac{\partial i}{\partial x} + \rho v \frac{\partial i}{\partial y} = \frac{1}{Pr}\frac{\partial}{\partial y}\left(\mu \frac{\partial i}{\partial y}\right) + (\varkappa - 1) M_1^2 \mu\left(\frac{\partial u}{\partial y}\right)^2,\\
&\rho i = 1, \quad \mu = i^n.
\end{aligned}\right\} \tag{8.67}$$

In the energy equation the symbol M_1 denotes Mach's number for the nonperturbed flow in the upper semiplane, i.e., the ratio between flow velocity u_1 and the speed of sound in it.

The boundary conditions for u and i in the range $x > 0$ in nondimensional variables take the form

$$\left.\begin{aligned}
u = 1, \quad i = 1 \text{ if } y = +\infty,\\
u = m_u, \quad i = m_i \text{ if } y = -\infty
\end{aligned}\right\} \tag{8.68}$$

(here $m_u = \frac{u_2}{u_1}$, $m_i = \frac{i_2}{i_1}$).

We assume, as above, $n = 1$, and use the transformation to Dorodnitsyn's variables

$$\xi = x, \quad \eta = \int_0^y \rho dy; \tag{8.69}$$

Equation (8.67) and the boundary conditions (8.68) then go over to the following equations:

$$\left.\begin{aligned} &u\frac{\partial u}{\partial \xi} + \tilde{v}\frac{\partial u}{\partial \eta} = \frac{\partial^2 u}{\partial \eta^2},\\ &\frac{\partial u}{\partial \xi} + \frac{\partial \tilde{v}}{\partial \eta} = 0,\\ &u\frac{\partial i}{\partial \xi} + \tilde{v}\frac{\partial i}{\partial \eta} = \frac{1}{Pr}\frac{\partial^2 i}{\partial \eta^2} + (\varkappa - 1) M_1^2 \left(\frac{\partial u}{\partial \eta}\right)^2, \end{aligned}\right\} \tag{8.70}$$

$$\left.\begin{aligned} &u = 1, \quad i = 1 \quad &\text{if } \eta = +\infty,\\ &u = m_u, \quad i = m_i \quad &\text{if } \eta = -\infty. \end{aligned}\right\} \tag{8.71}$$

In Eqs. (8.70) $\tilde{v} = \rho v + u\frac{\partial \eta}{\partial x}$.

In order to solve the latter system of equations we suppose that the longitudinal velocity component and the enthalpy are functions of a single variable:

$$u(\xi, \eta) = F'(\varphi_D), \quad i(\xi, \eta) = i(\varphi_D); \quad \varphi_D = \frac{\eta}{2\sqrt{\xi}}. \tag{8.72}$$

This problem can be reduced to integrating the following ordinary differential equations:

$$F''' + 2FF'' = 0, \tag{8.73}$$

$$i'' + 2Pr\,Fi' + Pr(\varkappa - 1) M_1^2 (F'')^2 = 0 \tag{8.74}$$

with the boundary conditions:

$$\left.\begin{aligned} &F' = 1, \quad i = 1 \quad &\text{if } \varphi_D = +\infty,\\ &F' = m_u, \quad i = m_i \quad &\text{if } \varphi_D = -\infty. \end{aligned}\right\} \tag{8.75}$$

Equation (8.73) and the corresponding boundary conditions (8.75) are analogous to the equivalent equation (5.23) and the boundary conditions (5.26) for the incompressible fluid. Consequently, all the results obtained in Section 5.3 from Eq. (5.23) will also apply to the case considered:

$$u = F'(\varphi_D) = 1 + \frac{1}{2}(m_u - 1)(1 - \operatorname{erf}\varphi_D),$$

$$\tilde{v} = \frac{1}{\sqrt{\xi}}(\varphi_D F' - F) = \frac{1}{2\sqrt{\xi}}(m_u - 1)\left\{\int_0^{\varphi_D}(\operatorname{erf} z)\,dz - \varphi_D \operatorname{erf}\varphi_D - C_1\right\}. \quad (8.76)$$

For the stream function we have

$$\psi(\xi, \eta) = 2\sqrt{\xi}\left\{\varphi_D + \frac{1}{2}(m_u - 1)\left[\varphi_D - \int_0^{\varphi_D}(\operatorname{erf} z)\,dz + C_1\right]\right\}. \quad (8.77)$$

As regards the general solution of the linear inhomogeneous differential equation (8.74) for the enthalpy, it can be represented as the sum of an arbitrary particular solution to Eq. (8.74) and the general solution of the corresponding homogeneous equation:

$$i_1'' + 2\Pr F i_1' = 0.$$

The latter equation has the solution (see Section 5.3).

$$i_1 = C_2 \int_0^{\varphi_D} [F''(z)]^{\Pr} dz + C_3.$$

As a particular solution of Eq. (8.74) we can take the function

$$J(\varphi) = -(\varkappa - 1) M_1^2 \int_0^{\varphi_D} [F''(z)]^{\Pr}\left(\int_0^z [F''(t)]^{2-\Pr} dt\right) dz.$$

The general solution of Eq. (8.74) satisfying the boundary conditions (8.75) has the following form:

$$i(\varphi_D) = 1 + \frac{1}{2}(m_i - 1)[1 - \operatorname{erf}(\varphi_D \sqrt{\Pr})] + J(\varphi_D) - J(\infty). \quad (8.78)$$

In the same form we would also have obtained the solution in the case of the incompressible fluid if we had taken the heat of friction into account in the initial energy equation. The two last terms in Expression (8.78) characterize the influence of the heat of friction on the enthalpy (temperature) distribution.

In order to estimate the character of motion of a gas and the temperature distribution in the mixing zone and to elucidate the influence of the flow parameters on the flow pattern, we must use the equations

linking the characteristic values of the quantity $\varphi_{u=0}$ and $\varphi_{\psi=0}$ with the parameter m_u. These equations obtained from the conditions $F' = 0$ and $\psi = 0$ with the use of Eqs. (8.76) and (8.77) have the form

$$-m_u = \frac{1 + \operatorname{erf} \varphi_{u=0}}{1 - \operatorname{erf} \varphi_{u=0}}, \tag{8.79}$$

$$m_u = \frac{C_1 - \varphi_{\psi=0} - \int_0^{\varphi_D} (\operatorname{erf} z)\, dz}{C_1 + \varphi_{\psi=0} - \int_0^{\varphi_D} (\operatorname{erf} z)\, dz}. \tag{8.80}$$

To return to the physical parameters we must use the equations linking the variables x, y with A.A. Dorodnitsyn's variables ξ, η:

$$x = \xi, \qquad \varphi = \frac{y}{2\sqrt{x}} = \int_0^{\varphi_D} i(z)\, dz. \tag{8.81}$$

Expressions (8.76) and (8.78) together with Eq. (8.81) yield a solution of the problem on the mixing of parallel and antiparallel flows of compressible gas.

On the basis of the results obtained we shall now determine the nature of the flow in the mixing zone and the influence of the parameters m_1 and M_1 on the velocity and temperature distribution.

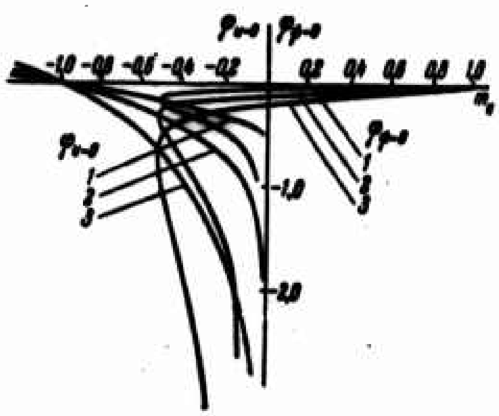

Fig. 8.8. The characteristic coordinates of the mixing zone of gas flows $(\varphi_{u=0}, \varphi_{\psi=0})$ as functions of the flow parameters. 1. $m_1 = 0.1$; 2. $m_1 = 1.0$; 3. $m_1 = 3.0$.

Figure 8.8 shows the curves illustrating the characteristic quan-

tities $\varphi_{u=0}$ and $\varphi_{\psi=0}$ as functions of the parameters m_u and m_i in the physical plane of x,y. The value of the constant C_1 in Eq. (8.80) was here taken equal to $C_1 = -0.05$. This fact does of course not influence the qualitative flow pattern with which we are here concerned. An elimination in the choice of the value of C_1 were only possible by way of a comparison of the theoretical and experimental results. Such experimental data for the laminar mixing of compressible gases are, however, not available.

It follows from Fig. 8.8 that with certain definite negative values of the parameter m_u an inverse motion of the gas must arise, the width and position of the zone of this inverse motion depending on the value of the parameter m_i. As m_i decreases this zone becomes narrower and shifts toward the line $y = 0$.

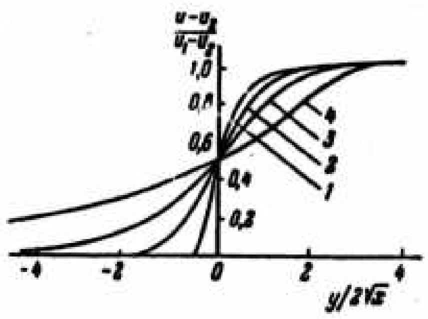

Fig. 8.9. Distributions of surplus velocity in the mixing zone of two gas flows.
1. $M_1 \approx 0$; $m_1 = 0{,}1$; 2. $M_1 \approx 1{,}0$; $m_1 = 1{,}0$; 3. $M_1 \approx 0$; $m_1 = 3{,}0$; 4. $M_1 = 1{,}5$; $m_1 = 3{,}0$.

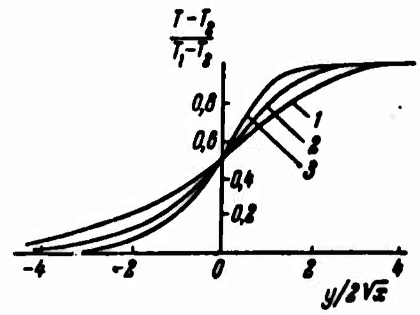

Fig. 8.10. Distributions of surplus temperature in the mixing zone of two gas flows.
($M_1 \approx 0$); 1. $Pr = 0{,}5$; 2. $Pr = 1{,}0$; 3. $Pr = 2{,}0$.

The influence of the parameters m_1 and M_1 on the distributions of the excessive nondimensional velocity $\frac{u-u_2}{u_1-u_2}$ is shown in Fig. 8.9. We see from the figure that the effective width of the mixing zone grows as the values of the parameters m_1 and M_1 increase.

Figure 8.10 illustrates the dependence of the temperature distributions on Prandtl's number. The nature of this dependence is the same

as in other cases of jet flows: the effective thickness of the thermal boundary layer increases as Pr decreases and vice versa.

Thus we have solved the problem of investigating the velocity and temperature distributions in the mixing zone arising in the interface of parallel or antiparallel flows and the influence on their flow parameters.

8.7. THE SEMILIMITED GAS JET

At the end of this chapter which has been devoted to the fundamental laws of the expansion of laminar jets of compressible gas, we consider one of the problems of semilimited gas jets, namely the problem of a jet expanding along a circular cone. It is of interest as it is one of the few examples of flows with axial symmetry to which A.A. Dorodnitsyn's transformation of variables can be applied successfully.

We maintain the position of the coordinate axes and the basic assumptions admitted when solving the problem of the semilimited jet of incompressible fluid (see Section 6.2): the source jet of compressible gas is slightly twisted. This permits an essential simplification of the compressible boundary layer equations which can be written in the following form [23,104]:

$$\left.\begin{aligned}
&\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}-\rho\frac{w^2}{x}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right),\\
&\rho\frac{w^2}{x}\operatorname{ctg}\omega=\frac{\partial p}{\partial y},\\
&\rho u\frac{\partial w}{\partial x}+\rho v\frac{\partial w}{\partial y}+\frac{uw}{x}=\frac{\partial}{\partial y}\left(\mu\frac{\partial w}{\partial y}\right),\\
&\frac{\partial}{\partial x}(\rho xu)+\frac{\partial}{\partial y}(\rho xv)=0,\\
&\rho u\frac{\partial i}{\partial x}+\rho v\frac{\partial i}{\partial y}=\frac{1}{Pr}\frac{\partial}{\partial y}\left(\mu\frac{\partial i}{\partial y}\right)+\mu\left(\frac{\partial u}{\partial y}\right)^2,\\
&p=\frac{\varkappa-1}{\varkappa}\rho i,\quad \mu=ci^n\quad (i=c_pT,\ c_p\approx\text{const}).
\end{aligned}\right\}\qquad(8.82)$$

The point is here (as also in the system of equations (6.47) for the incompressible fluid) that in the first equation of System (8.82) we can neglect the terms $-\rho\frac{w^2}{x}$ and $-\frac{\partial p}{\partial x}$ as they are small compared to the other terms of the equation. We shall see that this neglection is justi-

fied when we have obtained a solution which is valid at large distances from the source.

Starting from the estimations which are usual in the boundary layer theory, we find from the second equation of system (8.82) that the pressure change inside the boundary layer formed by the slightly twisted jet is of the order of the thickness of the boundary layer such that the equation of state can be rewritten in the form ip = const.

If in addition to this we assume $n = 1$ we can write the initial system of equations, instead of (8.82), in the form

$$\left.\begin{aligned}
&\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right), \quad \rho\frac{w^2}{x}\operatorname{ctg}\omega=\frac{\partial p}{\partial y},\\
&\rho u\frac{\partial w}{\partial x}+\rho v\frac{\partial w}{\partial y}+\rho\frac{uw}{x}=\frac{\partial}{\partial y}\left(\mu\frac{\partial w}{\partial y}\right),\\
&\frac{\partial}{\partial x}(\rho xu)+\frac{\partial}{\partial y}(\rho xv)=0,\\
&\rho u\frac{\partial i}{\partial x}+\rho v\frac{\partial i}{\partial y}=\frac{1}{Pr}\frac{\partial}{\partial y}\left(\mu\frac{\partial i}{\partial y}\right)+\mu\left(\frac{\partial u}{\partial y}\right)^2,\\
&\rho i=\rho_\infty i_\infty, \quad \frac{\mu}{\mu_\infty}=\frac{i}{i_\infty}
\end{aligned}\right\} \qquad (8.83)$$

(here, as usually, the symbol ∞ indicates that the respective quantity refers to the nonperturbed gas of the surrounding medium.

The system of equations (8.83) must be integrated with the following boundary conditions for the velocity components

$$\left.\begin{aligned}
&u=v=w=0 \quad \text{if} \quad y=0,\\
&u=w=0 \qquad \text{if} \quad y=\infty
\end{aligned}\right\} \qquad (8.84)$$

and the integral conditions of conservation

$$\left.\begin{aligned}
&\int_0^\infty \rho xu^2\left(\int_0^y \rho xu\,dy\right)dy=K=\text{const},\\
&\int_0^\infty \rho x^2uw\left(\int_0^y \rho xu\,dy\right)dy=N=\text{const},
\end{aligned}\right\} \qquad (8.85)$$

which are derived analogously as in Chapter 6.

As regards the boundary conditions and the integral invariants for the thermal problem we shall consider them in three variants, just as in the problem of the semilimited jet of incompressible fluid expanding along a cone.

a) The temperatures of the cone surface and the nonperturbed gas are the same:

$$\left.\begin{array}{c} \Delta i_0 = 0 \quad \text{if} \quad y = 0, \\ \Delta i_0 = 0 \quad \text{if} \quad y = \infty, \\ \int\limits_0^\infty \rho x u \, \Delta i_0 \left(\int\limits_0^y \rho x u \, dy \right) dy = K_T^1 = \text{const with } \Pr = 1 \\ \left(\Delta i_0 = i + \frac{u^2}{2} - i_\infty \right). \end{array}\right\} \qquad (8.86a)$$

In this case, we restrict ourselves to the simplest case of Pr = 1; the energy equation can then be written in the form

$$\rho u \frac{\partial \Delta i_0}{\partial x} + \rho v \frac{\partial \Delta i_0}{\partial y} = \frac{\partial}{\partial y} \left(\mu \frac{\partial \Delta i_0}{\partial y} \right). \qquad (8.87)$$

b) The surface of the cone is nonheatconducting:

$$\left.\begin{array}{c} \frac{\partial \Delta i}{\partial y} = 0 \quad \text{if} \quad y = 0, \\ \Delta i = 0 \quad \text{if} \quad y = \infty, \\ 2\pi \sin \omega \int\limits_0^\infty \rho x u \, \Delta i \, dy = Q = \text{const} \\ (\Delta i = i - i_\infty). \end{array}\right\} \qquad (8.86b)$$

c) The temperature of the cone surface is constant and different from that of the quiescent gas far away from the plate:

$$\left.\begin{array}{c} \Delta i = \Delta i_w \quad \text{if} \quad y = 0, \quad \Delta i = 0 \quad \text{if} \quad y = \infty \\ (\Delta i = i - i_\infty, \quad \Delta i_w = i_w - i_\infty). \end{array}\right\} \qquad (8.86c)$$

When solving the thermal problem with Conditions (8.86b) and (8.86c) we shall suppose that we can neglect the heat of friction in the energy equation. It can then be written in the form

$$\rho u \frac{\partial \Delta i}{\partial x} + \rho v \frac{\partial \Delta i}{\partial y} = \frac{1}{\Pr} \frac{\partial}{\partial y} \left(\mu \frac{\partial \Delta i}{\partial y} \right). \qquad (8.88)$$

Passing over to the plane of A.A. Dorodnitsyn's variables

$$\xi = x, \quad \eta = \int\limits_0^y \frac{\rho}{\rho_\infty} dy, \qquad (8.89)$$

we transform the initial system of equations to a form analogous to that of the corresponding system of equations of motion and continuity for a semilimited jet of incompressible fluid expanding along a cone:

$$\left.\begin{aligned}
&u\frac{\partial u}{\partial \xi}+\tilde{v}\frac{\partial u}{\partial \eta}=\nu_\infty\frac{\partial^2 u}{\partial \eta^2},\\
&\rho_\infty\frac{w^2}{\xi}\operatorname{ctg}\omega=\frac{\partial p}{\partial \eta},\\
&u\frac{\partial w}{\partial \xi}+\tilde{v}\frac{\partial w}{\partial \eta}+\frac{uw}{\xi}=\nu_\infty\frac{\partial^2 w}{\partial \eta^2},\\
&\frac{\partial}{\partial \xi}(xu)+\frac{\partial}{\partial \eta}(x\tilde{v})=0,
\end{aligned}\right\} \tag{8.90}$$

and the energy equation:

$$u\frac{\partial \Delta i_0}{\partial \xi}+\tilde{v}\frac{\partial \Delta i_0}{\partial \eta}=\nu_\infty\frac{\partial^2 \Delta i_0}{\partial \eta^2}, \tag{8.91}$$

or

$$u\frac{\partial \Delta i}{\partial \xi}+\tilde{v}\frac{\partial \Delta i}{\partial \eta}=a_\infty\frac{\partial^2 \Delta i}{\partial \eta^2}. \tag{8.92}$$

The boundary conditions and integral conditions can be written in the form

$$\left.\begin{aligned}
&u=\tilde{v}=w=0 \ \text{if} \ \eta=0,\\
&u=w=0 \ \text{if} \ \eta=\infty,\\
&\int_0^\infty \xi u^2\left(\int_0^\eta \xi u\,d\eta\right)d\eta=\frac{K}{\rho_\infty^2}=\text{const},\\
&\int_0^\infty \xi^2 uw\left(\int_0^\eta \xi u\,d\eta\right)d\eta=\frac{N}{\rho_\infty^2}=\text{const};
\end{aligned}\right\} \tag{8.93}$$

$$\left.\begin{aligned}
&\Delta i_0=0 \ \text{if} \ \eta=0,\\
&\Delta i_0=0 \ \text{if} \ \eta=\infty,\\
&\int_0^\infty \xi u\,\Delta i_0\left(\int_0^\eta \xi u_0' d\eta\right)d\eta=\frac{K_T}{\rho_\infty^2}=\text{const};
\end{aligned}\right\} \tag{8.94a}$$

$$\left.\begin{aligned}
&\frac{\partial \Delta i}{\partial \eta}=0 \ \text{if} \ \eta=0,\\
&\Delta i=0 \ \text{if} \ \eta=\infty,\\
&2\pi\rho_\infty\sin\omega\int_0^\infty \xi u\Delta i\,d\eta=Q=\text{const};
\end{aligned}\right\} \tag{8.94b}$$

$$\left.\begin{aligned}
&\Delta i=\Delta i_w \ \text{if} \ \eta=0,\\
&\Delta i=0 \ \text{if} \ \eta=\infty.
\end{aligned}\right\} \tag{8.94c}$$

Examples of calculating the velocity distributions in a semilimited gas jet and the temperature distributions (for three forms of boundary conditions) are shown in Figs. 811-8.13 which clearly illustrate the influence of compressibility.

A solution to the system of equations analogous to (8.90)-(8.92) with Conditions (8.93)-(8.94c) has been obtained above (see Chapter 6).

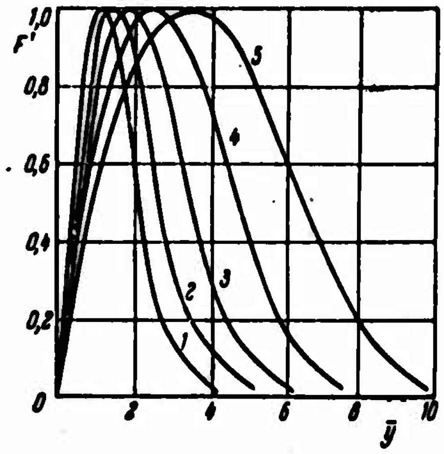

Fig. 8.11. Velocity distributions in a semilimited gas jet. ($T_w = T_\infty$; $Pr = 1{,}0$; $\bar{x} = 1$). 1. $\bar{M} \approx 0$; 2. $\bar{M} = 0{,}5$; 3. $\bar{M} = 1{,}0$; 4. $\bar{M} = 1{,}5$; 5. $\bar{M} = 2{,}0$ ($\bar{M} = K_T/K\sqrt{c_p T_\infty}$).

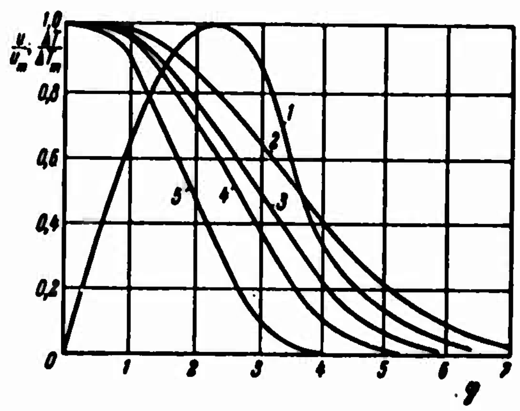

Fig. 8.12. Velocity and temperature distributions in semilimited gas jet on heat-insulated plate ($\bar{M} = 0$; $T_w/T_\infty = 2$ in conventional initial cross section). 1) Velocity (Pr = 0.75); 2-5) temperature; 2. $Pr = 0{,}5$; 3. $Pr = 0{,}75$; 4. $Pr = 1{,}0$; 5. $Pr = 2{,}0$.

As regards the return to the physical plane of coordinates x and y it is carried out in the same way as in the previous problem of the plane laminar gas jet.

The calculations of laminar semilimited jets of compressible gas considered in this chapter can be expanded to the case of chemical reactions (among them dissociation, recombination and the like and also

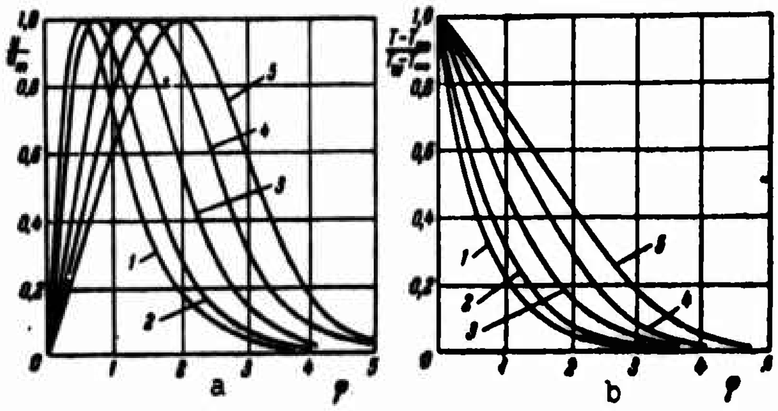

Fig. 8.13. Velocity (a) and temperature (b) distributions in semilimited gas jet along plate of constant surface temperature.
($\bar{M} = 0$; $Pr = 0.75$). 1. $T_w/T_\infty = 0.25$; 2. $T_w/T_\infty = 0.5$; 3. $T_w/T_\infty = 1.0$; 4. $T_w/T_\infty = 1.5$; 5. $T_w/T_\infty = 2.0$.

combustion reactions) taking place in the gas volume or at the surface of the body placed in the flow. A problem of this kind, however, goes beyond the limits of this book. We shall therefore restrict ourselves to referring the reader to a solution for a semilimited gas jet reacting at the surface of a cone (see [71]). This paper, as also other analogous papers, may serve as an example of some kind of bridge between problems of the theory of jets on the one hand and the theory of heterogeneous combustion, on the other. A feature which is particularly characteristic of them is the presence of two quite different courses of the process (before and after inflammation) and the critical phenomena of transition between them.

LITERATURE REFERENCES

23, 25, 32, 47, 49, 58, 71, 74, 92, 104, 114, 117, 118, 119, 120, 134, 135, 160, 175, 212, 256, 257, 267, 268, 271, 272, 279, 318, 324.

Manuscript Page No.	Footnote
182	In a circular motion, the influence of the Prandtl number Pr in the energy-redistribution effect recedes into the background (see [193]).

Part Three

TURBULENT JETS OF FLUIDS AND GASES

This part of the book is devoted to turbulent flows of viscous fluids and gases. Owing to the absence of exact equations of turbulent flow we must treat the problem in a way different from that in the previous chapters. As an initial qualitative model of concrete turbulent flow we use the solution of an analogous problem of the theory of laminar jets (if it is known). Thus, for self-similar turbulent jets of incompressible fluid, we can use all the equations (in nondimensional form) and their solutions obtained by the method of the asymptotic boundary layer. The values of the constants will of course be different. Detailed data for such problems will be given in Table 11.1.

The experiment is, however, of fundamental significance for the development of the procedures of calculation in the theory of turbulent jets. In the first chapters we shall therefore briefly recall the basic experimental data on the expansion of turbulent jets. We shall also give the equations of the turbulent boundary layer. In this connection, we shall discuss the various methods of calculation. In particular, great attention will be paid to the so-called turbulent Prandtl number.

A separate chapter is devoted to the expansion of a turbulent (semilimited) jet of incompressible fluid along a solid surface. Experimental data proving that the boundary layer of a jet becomes turbulent near the wall are discussed in detail.

When calculating turbulent gas jets our main attention is paid to the turbulent mixing in compressible gas and the calculation of self-si ilar gas jets on the basis of the supposition of similar momentum flux density (ρu^2) distributions.

A wider method of calculation of the theory of turbulent jets, the method of the equivalent problem of the theory of heat conduction, is partially connected with the latter problem. We shall discuss in detail the fundamentals and the development of this method which consists of a unique semiempirical linearization of the equations of a free turbulent boundary layer of fluid or gas. We shall describe the procedure of applying it to various problems. With the help of this method (when solving the "inverse problem") new formulas are obtained for the calculation of turbulent friction and heat conduction. It is a peculiarity of these formulas, which in the simplest cases take an intermediate position between the well-known "old" Prandtl formulas and the "new" ones, that the expression for, e.g., the turbulent stress of friction, contains two terms.

In the last chapter of this part of the book, we calculate the turbulent gas jets on the basis of the supposition on the similarity of ρu^2 and mainly by the method of the equivalent problem. Here we give a detailed comparison of the results of calculation and the experimental data on the expansion of various (plane, axisymmetric, submerged, parallel, etc.) turbulent jets of fluid or gas, among them also non-self-similar ones.

Chapter 9

EXPANSION OF TURBULENT JETS

9.1 THE GENERAL LAWS

An investigation of turbulent jets must be based on both the qualitative flow pattern taken from the theory of laminar jets and the fundamental laws governing the expansion of such flows observed in experiments. At present, this way of treating the problem is the only one which is correct as we have no closed system of physically strict and well-established equations of turbulent motion at our disposal. The most advanced modern theory of turbulence, the statistical theory developed in the papers by A.N. Kolmogorov, L.D. Landau, A.M. Obukhov, W. Heisenberg, G. Bachelor, et al., does not dispose so far of sufficient data for a calculation of anisotropic turbulent motions (in particular, turbulent jets), without using empirical constants. Taking into account the complexity and cumbrousness of the apparatus of the statistical theory of turbulence [181,200] it is, as already mentioned, expedient to base our considerations for the present on the empirical flow pattern and the generalized, unexact but illustrative so-called "semiempirical" methods of calculation which, in addition to this, are easy to apply. Many of them have been developed in analogy to the methods we discussed in detail in the previous parts of the book for the laminar jets. In addition to this, not only individual methods of calculation but also the fundamental physical laws permitting the consideration of the problem of the expansion of turbulent

jets as a separate and independent class of motions of viscous fluids of gases again display to some degree (of course in another quantitative aspect) the fundamental properties of laminar jets.

In fact, when we consider the series of graphs represented in Figs. 9.1-9.5 taken from various experimental papers, we find many results which agree qualitatively with the analogous results obtained for laminar jets. First of all, this holds true for the so-called similarity of the velocity distributions and those of the temperature difference in the cross sections of turbulent jets of incompressible fluids, or the quantity ρu^2 in gas jets. This similarity or, what is the same, the universality of the nondimensional distributions is the same for the characteristic regions of the flow within a wide range of values of Reynolds' number calculated from the efflux parameters of the jet. This enables us to speak of a self-similar flow in a more general sen than in the case of laminar jets. In laminar jets the transverse velocity distribution in a source jet, given in the coordinates $\frac{u}{u_m} = f(\varphi)$ was also independent of the Reynolds number. The absolute position of a concrete distribution (i.e., the transition from the power function φ to the coordinate x) was, however, connected with the value of Re that entered the variable φ (for example, at the boundary of a plane jet where $\varphi = y\sqrt{\frac{u_0}{\nu x}}$ etc.). In contrast to this, the expansion of a turbulent jet is, as shown by experiment, virtually independent of the coefficient of viscosity of the fluid*, etc.

It must be noted that the relative distributions $\frac{u_m}{u_0} = F\left(\frac{x}{d}\right)$ and the analogous ones along the jet obtained in experiments with various arrangements and drawn in real (nontransformed) geometrical coordinates are, as a rule, not in agreement with one another. This difference is noticed not only near the nozzle issuing the jet where the influence of the initial conditions is great, but also away from it. It is ex-

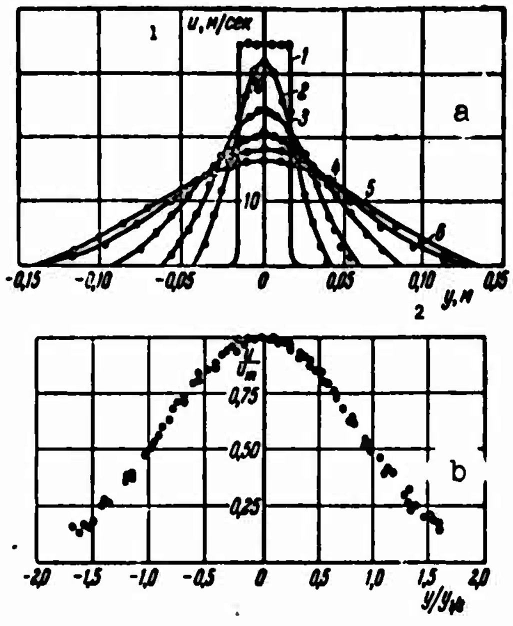

Fig. 9.1 Velocity distributions in the plane of the jet (E. Förtmann's data [244]). a) Distributions in different cross sections.

№	1	2	3	4	5	6
x, м	0	0.2	0.35	0.50	0.625	0.75

b) Universal distributions ($x = 0.2$–$0.75m$). 1) u, m/sec; 2) y, m.

plained by the finer properties of the jet's hydrodynamic structure (level of pulsation, etc.). Influencing the latter (by way of nozzle profiling, arrangement of grids and the like) the intensity drop along the jet axis can be varied within well-known limits, the general character of the flow is, however, maintained in this case.

From the other peculiarities we see that for all free turbulent jets the effective boundaries of the source jet are rectilinear and that the thermal boundaries of the jet are always a little broader than the dynamic ones. This fact indicates that the effective coefficient o turbulent heat transfer (a_T) is always greater than the effective coefficient of turbulent momentum transfer (v_T). In other words, the ratio of these coefficients which is usually called the "turbulent Prandtl number" $Pr_T = \frac{v_T}{a_T}$, is always smaller than unity. Disregarding the

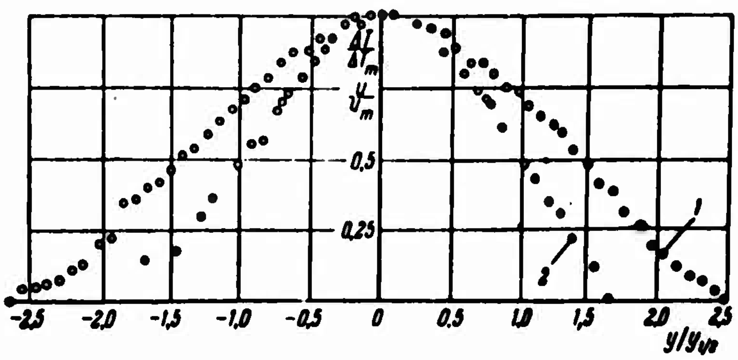

Fig. 9.2 Universal temperature and velocity distributions in a plane jet (after data by H. Reichhardt [2,2]). 1) Temperature; 2) velocity.

dependence on the values of the true physical constants of the medium, the turbulent Prandtl number is approximately equal to about 0.7-0.8 (i.e., close to the physical Prandtl number for air).

Another fact worth noting, a property possessed by all turbulent jets: the distributions of the characteristic quantities (velocity, temperature, etc.), far away from the nozzle issuing the jet, display the same universal character, regardless of the dependence on the initial conditions of efflux. Thus, for example, at a distance of the order of 20 or more nozzle diameters, the distribution of the velocity u/u_m in the jet cross section will be the same for the efflux from a round, square or triangular nozzle. This fact, which is established experimentally, shows that the conception of the source jet is fully suitable to describe the laws of expansion of turbulent jets far away from the nozzle.

As already mentioned in the introduction, the property of self-similarity of a motion is in its physical mechanism akin to the so-called regularization of the temperature distribution in processes of unsteady heat conduction. It is interesting not only as a proof of the

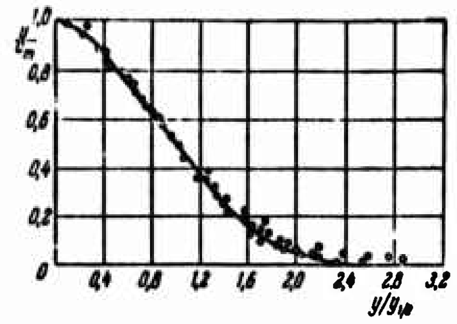

Fig. 9.3. Universal velocity distribution in round jet ($x/d = 12 \div 30$; $M = 0.1 \div 0.5$); (after data from [260]).

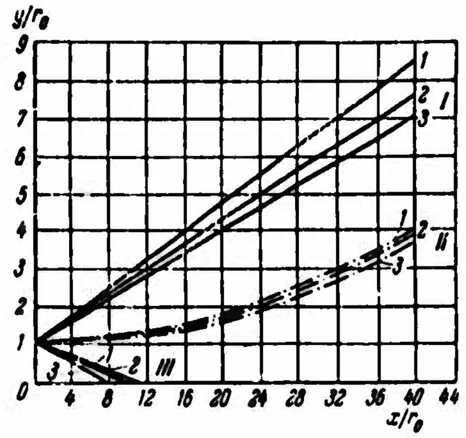

Fig. 9.4. "Topography" of round jet (after data by I. Laurence [260]. 1. $M = 0.3$; 2. $M = 0.5$; 3. $M = 0.7$; I — $(u/u_m) \approx 0$; II — $(u/u_m) = 1/2$; III — $(u/u_m) = 1.0$.

similarity of the leveling processes of momentum and the like in turbulent flows on the one hand and the heat conduction on the other, but also, from the mathematical point of view, as it is p·ssible to use the equations of self-similar flow in the theoretical description of the process.

We also want to indicate that experimental data on gas jets, in agreement with those given in Fig. 8.2, speak in favor of a faster damping of ve..city (and surplus temperature) in the case where the jet expands in a denser medium (e.g., hot jet of gas in cold air), and vice

versa. As established in a series of papers [33, 50, 52, 58, 114, etc.] in jets of compressible gas the assumption of a similarity of the distributions of velocity and, particularly, of surplus temperature in the self-similar section of a jet is rather far from being real. The assumption of similarity of the distributions of "transferred substances," momentum flux density ρu^2, flux density of excessive heat content $\rho c_p u \Delta T$ etc., is in much better agreement with the experiments.

As already mentioned, the influence of the parameter of compressibility (the ratio of gas density in the surrounding medium to density in the jet, $\omega = \frac{\rho_\infty}{\rho_0}$) on the distributions of velocity and surplus temperature is qualitatively the same as in laminar jets (see Fig. 8.2). This proves once more the close relationship between the theories of laminar and turbulent jets. For the latter, however, $Pr_T < 1$. In the thermal problem of the turbulent jet of fluid or gas it has therefore become unnecessary to study the influence of Prandtl's number on the temperature

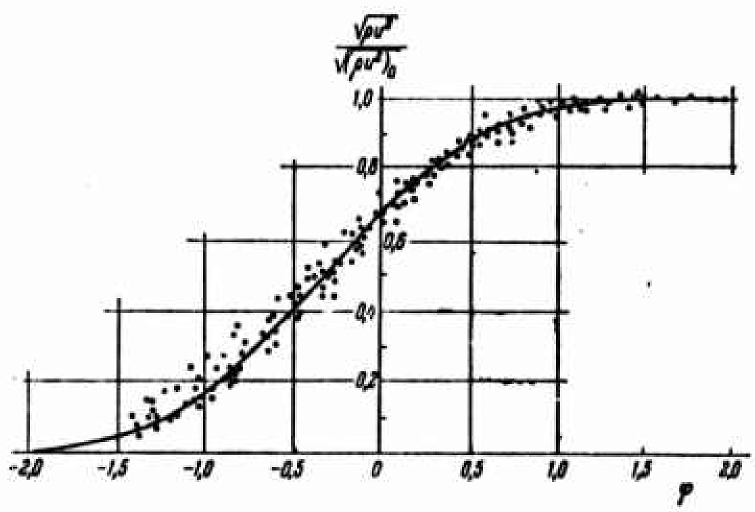

Fig. 9.5. The universal dynamic pressure distribution for the jet boundary (after data by Sh.A. Yershin and Z.B. Sakipov [96]) with $\omega = \rho_\infty/\rho_0 = 1$-14.5; isothermal air jet, $\omega = 1$; jet of combustion products, $\omega = 4$-5.3; efflux of hydrogen in air, $\omega = 14.5$.

distribution, etc. Taking this difference into account, the solutions obtained for laminar jets may serve as a model for the turbulent ones.

More than that, for self-similar jets, the nondimensional equations and solutions are identical (with ρ const).

9.2 THE MICROSTRUCTURE OF TURBULENT JETS

A detailed investigation of the structure of pulsations in free turbulent jets is beyond the framework of this book. We only give some results of experimental investigations of the field of the pulsation characteristics of turbulent jets. A series of examples, taken from experimental papers dealing with the study of the microstructure of turbulent jets [20,238,239,255,260,etc.] are shown in Figs. 9.6-9.10. Most of them were obtained by thermoanemometrical measurements in air jets. Referring the reader who is interested in details of the experiment to the literature mentioned above*, we shall focus our attention to some qualitative results.

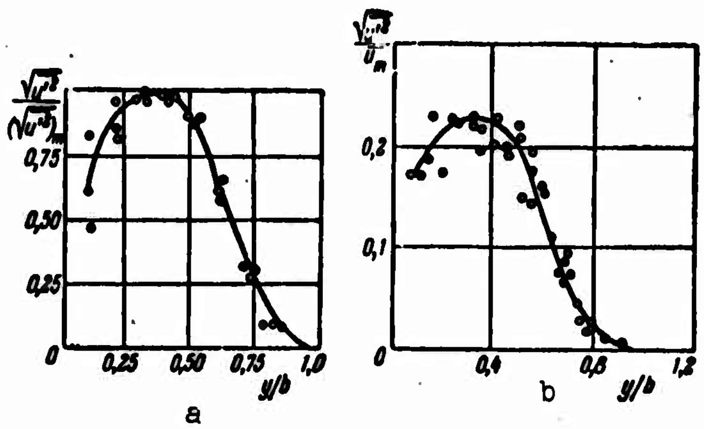

Fig. 9.6. Nondimensional distribution of velocity pulsation in round jet (x/d = = 7.5-10.0 after data by G.S. Antonova [11,20]).

$\frac{x}{d_0}$	$\frac{y}{r_0}$	$\frac{\sqrt{\overline{u'^2}}}{u_{яок}}$, %	$\frac{\sqrt{\overline{v'^2}}}{u_{яок}}$, %
3	1	24,1	24,2
3	0	10,1	11,8
4	1	26,6	24,1
4	0	14,8	13,4

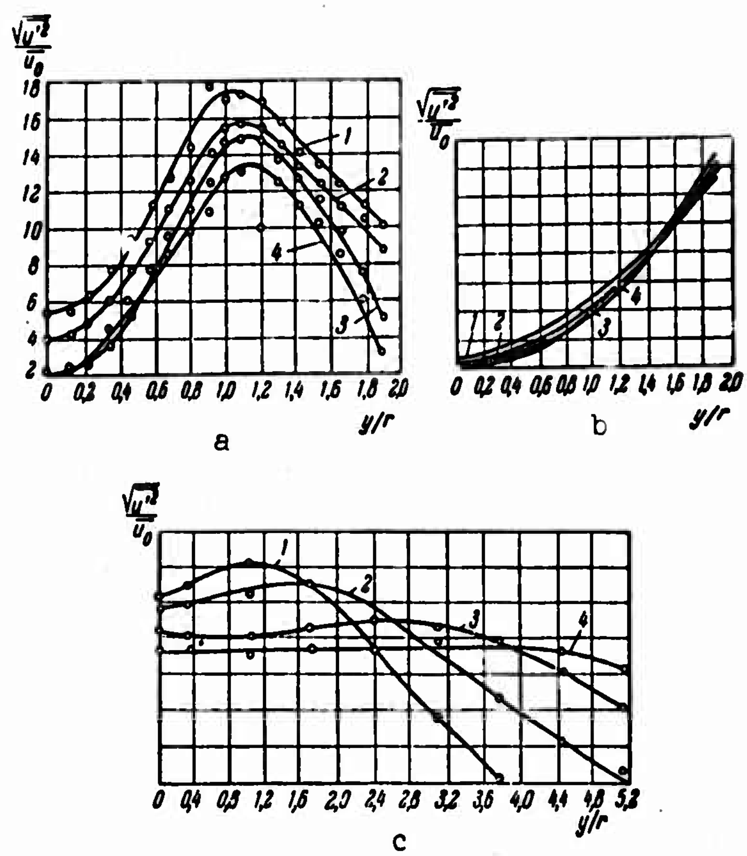

Fig. 9.7. Distributions of velocity pulsation in cross sections of a round jet (after data by I. Laurence [260]).

a) b) } $\frac{x}{d} = 3{,}8$

№	1	2	3	4
M	0,2	0,3	0,5	0,7
$Re \cdot 10^{-3}$	192	300	500	725

c) $M = 0{,}3$; $Re = 3 \cdot 10^{5}$

№	1	2	3	4
x/d	8	12	16	20

The most important results are due to the presence of a rather clear almost direct proportionality between the intensity of pulsation of the velocity vector and the derivative of the mean velocity with res-

pect to the normal to the direction of the jet. This connection (also known for streams in channels and the like [181]) shows that L. Prandtl's empirical formula $u' = l\frac{\partial u}{\partial y}$ (l is the mixing length) in a first approximation corresponds to the true flow pattern.

In a sufficiently good approximation we also have proofs of the proportionality between the pulsations of the longitudinal and the transverse velocity components. To illustrate this we give a table with data for an axisymmetric air jet. A comparison between the intensities of pulsation of u' and v' may be found in paper [260].

As regards the absolute level of pulsation, it reaches in a free jet a value of the order of twenty per cent (of the local magnitude of the velocity in the middle part of the jet), i.e., it is higher by almost one order of magnitude than for flows in smooth tubes. The pulsation intensity, which increases along the jet, passes through a maximum and then drops; in the cross sections the maximum of velocity pulsation coincides approximately with the inflexion of the mean velocity distribution curve.

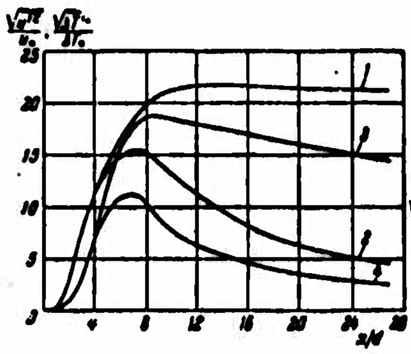

Fig. 9.8. Variation of intensity of velocity and temperature pulsation along the jet axis (after data by S. Corrsin and M. Uberoi [259]). 1-2) Pulsation of longitudinal velocity component referred to mean value of axial velocity (1) and initial velocity (2); 3-4) temperature pulsation, referred to surplus temperature on the axis (3) and in the outlet cross section (4).

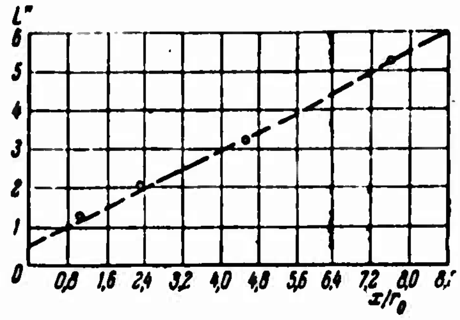

Fig. 9.9. Transverse scale unit of turbulence as a function of the distance (after data by I. Laurence [260]). $y/r_0 = 1$; M = 0.3; $Re_0 = 3.10^5$.

In the self-similar sections of the jet, we may also observe an approximate similarity of the distributions of pulsation characteristics (the rms pulsation and, with more spread, the values of the correlation coefficients).

Another feature whose experimental verification is of principal significance is the linear dependence on the coordinate x (in self-similar sections) of the transverse scale unit of turbulence, determined by the formula $L = \int_0^\infty R_{u_1u_2}\,dy$, where $R_{u_1u_2} = \frac{\overline{u_1'u_2'}}{\sqrt{\overline{u_1'^2}}\sqrt{\overline{u_2'^2}}}$ is the correlation coefficient between the pulsating velocities u_1' and u_2' at two points of the jet (in one and the same cross section). This observation shows that in a turbulent jet there exists in fact a statistical characteristic parameter of the dimension of a length, a scale unit, introduced as the Prandtl mixing length, L = cx.

It is also of interest to consider the variation of the frequency spectrum of the turbulent pulsations in a jet. We may see from a typical diagram of the spectral density of pulsation [260] that the main energy of pulsation, according to a rough estimation, must be in the range from 100 to 600-700 hc, while the whole spectrum covers a range from about 20 hc up to several thousands.

Measurements carried out with the help of frequency filters and also visual observations by means of a cathode oscillograph shows that the frequency of pulsation drops considerably as we approach the outer effective boundary of the jet. In the direction of the ordinate corresponding to half the maximum velocity value and, in particular, near the boundary of the jet, we may observe with the help of an oscillograph the phenomenon of the so-called "intermittence" of turbulence [25,181,193,206, etc.], that is, an alternate (in time) succession of periods of virtually laminar flow patterns and such of turbulent ones. This fact might yield an explanation of the disagreement of the mean velocity distributions in jet cross sections near the jet boundaries observed when comparing theoretical and experimental results, which is caused by an alternation of laminar and turbulent "viscosity." The influence of this intermittence on the structure of the turbulent jet (in analogy to the problem of the wake [181,229]) has at present not yet been taken into account comprehensively though it would be of great interest.

We also know [255] that in the case of the expansion of a jet in a wake the level of turbulent velocity pulsation is much lower than in a free jet. This is explained by assuming the parallel flow to exert a "pressing" effect.

As regards the pulsation of temperature, it is (see the data by G.S. Antonova [11,204] and [239]) much smaller than the velocity pulsation (see Fig. 9.8) but it resembles the latter as to order of magnitude and characteristic features. The differences in the u' and T' pulsation intensities make it very difficult to separate in thermoanemometrical measurements the temperature pulsations from the general indications, i.e., the pulsations of the electrical tension on the wires of the thermoanemometer which react upon both the velocity and the temperature pulsations at the same time. Such a separation was

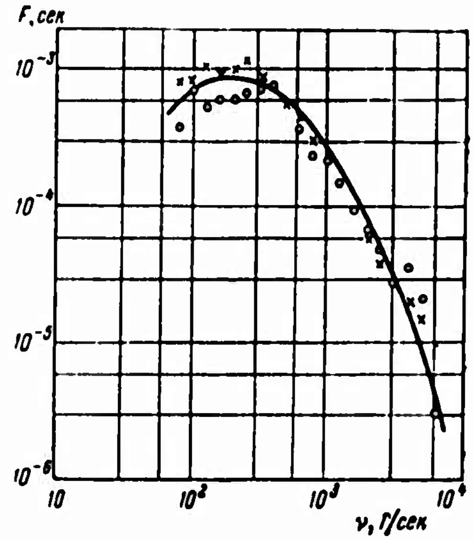

Fig. 9.10. Typical curve of spectral density of turbulent pulsations (after data by I. Laurence [260]). o o o o $y/r_0 = 0$; $M = 0.3$; $Re_0 = 3\cdot10^5$; $x/d = 6 \div 8$. × × × × $y/r_0 = 1$.

carried out independently in the papers by S. Corrsin and M. Uberoi [239] and G.S. Antonova, L.A. Vulis and P.V. Chebyshev in 1949.

Apart from the methodical significance, it is interesting for the understanding of the nature of these pulsations. On the basis of simple considerations on the origin of the pulsations as the displacement of various jets of fluid it must be assumed that in the air jet which is hot compared to the surrounding medium where velocity and temperature are maximum at the jet axis, the correlation between u' and T' must be positive. In the opposite case, in a cold air jet, compared to the surrounding medium, where at the jet axis the velocity is maximum but the temperature minimum, we must expect that the correlation between u' and T' is negative. In fact, in the first case (Fig. 9.11a) the air stream hitting the anemometer wire from a central zone of the jet carries along particles of higher velocity and higher temperature and the stream from the periphery such of smaller velocity and lower temperature.

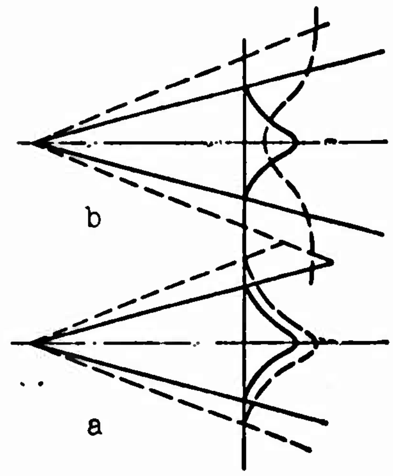

Fig. 9.11. Schematic representation of nonisothermal jet. a) $R_{uT} > 0$; b) $R_{uT} < 0$; ———— velocity; ---------- temperature.

In the second case (Fig. 9.11b) the wire is hit by an air stream of higher velocity and lower temperature (from the center) or, vice versa, of lower velocity and higher temperature (from the periphery).*

This result, the dependence of the sign of the $\overline{u'T'}$ correlation on the relation between the velocity and temperature distributions, has been verified by experiments. Calculation shows that the sign of the correlation is connected with the shape of the curve showing the voltage pulsation intensity on the wire as a function of the temperature difference between wire and air the wire being warmer than the air (if $R_{uT}=\frac{\overline{u'T'}}{\sqrt{\overline{u'^2}}\sqrt{\overline{T'^2}}}>0$ this curve is an ellipse, if $R_{uT}<0$ it is a parabola). This has also been proven by experiment. Analogous considerations were also applied (and verified by experiment) to the dependence of the sign of the correlations of the u' and v' pulsations on the distributions of the mean values of u and v.

The measurements of the pulsation structure of turbulent jets

show in this way that the averaged pulsation characteristics $\sqrt{\overline{u'^2}}$, $\sqrt{\overline{v'^2}}$, $\overline{u'v'}$, $\sqrt{\overline{T'^2}}$, $\overline{u'T'}$, L etc., are simply linked with the mean characteristics of the flow. This fact indicates that it is promising to develop the methods of the statistical theory of turbulence and that it is possible to use them even in near future in practical applications of the theory of turbulence of jets. At present, we may only point out the importance of systematic measurements of the pulsations, of more profound interpretations of the results of measurements of the mean values and of the collecting of data for a further development of the theory.

9.3. THE EQUATIONS OF THE TURBULENT BOUNDARY LAYER

Turbulent boundary layer equations written in terms of the mean values can be obtained on one of the following ways: by a transition from the Navier-Stokes equations to the Reynolds equations and a subsequent estimation of the terms of these equations, or by introducing the mean and pulsatory quantities in the Prandtl equation of the laminar boundary layer [212]. Let us briefly recall the second way of derivation, first considering a jet flow of incompressible fluid.

We go back to the initial system of equations, (4.3), which, when we use the continuity equation, can be rewritten in the following form:

$$\left.\begin{aligned}
&\rho\left[\frac{\partial u^2}{\partial x}+\frac{1}{y^k}\frac{\partial}{\partial y}(y^k uv)\right]=-\frac{\partial p}{\partial x}+\mu\frac{1}{y^k}\frac{\partial}{\partial y}\left(y^k\frac{\partial u}{\partial y}\right),\\
&\rho\frac{\widetilde{w}^2}{y}=\frac{\partial p}{\partial y}\qquad(\widetilde{w}=kw),\\
&\rho\left[\frac{\partial}{\partial x}(u\widetilde{w})+\frac{1}{y^{2k}}\frac{\partial}{\partial y}(y^{2k}v\widetilde{w})\right]=\mu\frac{1}{y^{2k}}\frac{\partial}{\partial y}\left\{y^{2k}\left[\frac{\partial\widetilde{w}}{\partial y}-\frac{\widetilde{w}}{y}\right]\right\},\\
&\frac{\partial}{\partial x}(y^k u)+\frac{\partial}{\partial y}(y^k v)=0,\\
&\rho c_p\left[\frac{\partial}{\partial x}(uT)+\frac{1}{y^k}\frac{\partial}{\partial y}(y^k vT)\right]=\lambda\frac{1}{y^k}\frac{\partial}{\partial y}\left(y^k\frac{\partial T}{\partial y}\right).
\end{aligned}\right\}\quad(9.1)$$

Remember that these equations are written for a plane (k = 0) or axisymmetric (k = 1) motion.

In the system of equations (9.1) we replace all the actual quantities (u, v, w, p, T) by the sums of the mean and the pulsatory quantities such that

$$u = \bar{u} + u', \quad v = \bar{v} + v' \text{ etc.}$$

(where, as usual, $\overline{u'} = \overline{v'} = 0$, $\overline{u^2} = \bar{u}^2 + \overline{u'^2}$, $\overline{uv} = \bar{u}\bar{v} + \overline{u'v'}$ etc.). The physical constants of the medium (μ, λ, c_p) are assumed constant, for the sake of simplicity.

In the final form of the equations of the averaged quantities we take into account that the averaged motion is steady, the effects of molecular momentum and heat exchange are negligibly small compared to the turbulent pulsations) and, finally, we assume that $\frac{\partial}{\partial y} \gg \frac{\partial}{\partial x}$. as usually in the boundary layer theory.

Under these conditions (after a repeated application of the continuity equation) the equations can be written in the form

$$\left.\begin{aligned}
&\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right) = -\frac{\partial \bar{p}}{\partial x} + \frac{1}{y^k}\frac{\partial}{\partial y}(y^k \tau_{xy}),\\
&\rho\frac{\overline{\tilde{w}^2}}{y} = \frac{\partial \bar{p}}{\partial y},\\
&\rho\left[\bar{u}\frac{\partial \overline{\tilde{w}}}{\partial x} + \bar{v}\frac{\partial \overline{\tilde{w}}}{\partial y} + \frac{\bar{v}\overline{\tilde{w}}}{y}\right] = \frac{1}{y^k}\frac{\partial}{\partial y}(y^k \tau_{y\varphi}),\\
&\frac{\partial}{\partial x}(y^k \bar{u}) + \frac{\partial}{\partial y}(y^k \bar{v}) = 0,\\
&\rho c_p\left(\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y}\right) = \frac{1}{y^k}\frac{\partial}{\partial y}(y^k q).
\end{aligned}\right\} \qquad (9.2)$$

The quantities of the turbulent tangential stresses of friction and heat flux which enter Eq. (9.2) are determined by the formulas

$$\tau_{xy} = -\rho\overline{u'v'}, \quad \tau_{y\varphi} = -\rho\overline{v'\tilde{w}'}, \quad q = -\rho c_p \overline{v'T'}.$$

The further way of solution consists of a choice of a concrete form of the dependence of the tangential frictional stresses and the heat flux on the coordinates and the mean values of velocity, temperature and their derivatives, on the basis of some intuitive considerations verified by experiment. There exist, generally speaking, various methods for such a choice. We can write immediately the sought expres-

sions for τ and q or, introducing the coefficients of turbulent transfer in their original forms, in analogy to the laws of molecular friction and heat conduction:

$$\tau_{xy} = \mu_T \frac{\partial \bar{u}}{\partial y}, \quad \tau_{y\varphi} = \mu_T \left(\frac{\partial \bar{\bar{w}}}{\partial y} - \frac{\bar{\bar{w}}}{y}\right), \quad q = \lambda_T \frac{\partial T}{\partial y}. \qquad (9.3)$$

In the latter case, which goes back to Bussineskiy (1877), the determination depends on the form of the expressions for the coefficients of turbulent exchange, μ_t and λ_t or $\nu_T = \frac{\mu_T}{\rho}$ and $a_T = \frac{\lambda_T}{\rho c_p}$. Here we may speak also formally of a turbulent analogue of the Prandtl number equal to the ratio ν_T/a_T (briefly the turbulent Prandtl number). Note however that for an incompressible fluid the transition to formulas of the type of Newton's law for the internal friction or Fourier's law for the heat conduction is in some way justified by the fact that the molar transfer of momentum and heat is observed in the case of the presence of a mean-velocity gradient and a mean-temperature gradient. In the case of a compressible gas and a variable density distribution this correspondence may be violated. The concrete form of the expressions for the coefficients of turbulent exchange (or turbulent friction and heat conduction) used when investigating jet flows of incompressible fluid will be discussed in the following section. Here we restrict ourselves to indicating that, with the help of experimental data, it is possible to calculate some self-similar turbulent jet flows of incompressible fluid within the framework of semiempirical methods. In other cases (e.g., a semilimited turbulent jet) such a calculation is more difficult and can be carried out by way of a "junction" of expressions applying to free and close-to-the wall turbulent boundary layers.

These methods of solution prove inapplicable or very complex and cumbersome in the case of many nonself-similar jet flows of incompressible fluids. If we consider, for example, such a problem of practical

importance as the expansion of a turbulent jet issued from a nozzle of finite dimensions, it has so far not been possible to solve it effectively by means of some direct semiempirical method. In these cases in engineering calculations we must have recourse to the relatively complex methods of constructing the jet in separate sections (initial, intermediate and final sections, see [4]), to an immediate choice of the velocity and temperature distributions, for example, according to the well-known formula by H. Schlichting (see [11]), or we assume them given in the form of polynomials and the like in the method of integral relations.

Yet more complex is the case of turbulent jets of compressible gas. In this connection the following must be said. If we apply systematically Reynolds' method of the representation of all actual quantities (velocity components, parameters of state, physical constants, etc.) in the form of sums of averaged and pulsatory components, we obtain, after having passed over in the boundary layer equations to the averaged quantities, a great number of terms containing mean products of pulsations of the velocity components, the density, etc.

The experimental data available at present are insufficient for a well-founded choice of semi-empirical expressions permitting the final formulation in the differential equations of at least a single averaged quantity (without introducing a great number of empirical constants).

Thus, as shown e.g., by L. Howard [175], a decomposition of all quantities into averaged and pulsatory components, though it enables us to write equations, does not show a way of applying simple intuitive formulas permitting an effective solution of problems of turbulent flows of incompressible fluids. Therefore, in a great number of papers, one has introduced some restricting assumptions, e.g., not only the mean pressure but also the actual pressure are supposed constant which

permits the elimination of density pulsations, such pulsations are canceled immediately, triple products of pulsatory quantities are neglected and the like (see [2, 4, 48 etc.]).

The small effectiveness of such assumptions is connected with the fact that it is impossible to know beforehand the limits of their applicability to turbulent motions of compressible gas (limited velocity, temperature drop, etc.).

Another way of calculation encountered in papers on the theory of turbulent jets of compressible gas is based on the fact that a decomposition of the form $\Phi = \overline{\Phi} + \Phi'$ applies to both the "original" quantities (u, v, p, and T, etc.) and the "complex" ones, the flux density ρu of the fluid [41], the momentum flux density $\rho u^2$and the flux density of excessive heat content $\rho c_p u \Delta T$ [50, 55], the stagnation temperature $T_0 = T + u^2/2c_p$ etc. The usefulness of each of these assumptions separately may, of course, not be estimated immediately by comparing the final results of the calculations with those of experiments. In addition to this we have at present only insufficiently reliable experimental data on compressible gas jets at our disposal. A calculation of such jets by semiempirical methods derived from the classical procedures by Prandtl for incompressible fluids (the mixing length, etc.) is therefore connected with great difficulties. This may be explained by the fact that for an extrapolation of the method of calculation to the region of compressible gas jets the single (necessary) condition of the return at ρ = const is unsufficient for a justification of the mathematical method of treating turbulent jets of incompressible fluid. A series of attempts (not yet successful, of course) of developing a procedure of calculation of compressible gas jets are contained in the papers by G.N. Abramovich [7, 8, 11, 12 etc.]. We do not intend to consider them here but, at the end of this part of the book, we want to discuss the

calculation according to the method of the equivalent problem in the theory of heat conduction which, in the authors' opinion, is most promising.

A weak point of virtually all methods of calculating turbulent jets of compressible gas (and the comparison of the theoretical results with experimental data) is the direct or indirect ignoring of the difference between the mean values of composite quantities and the products of means. For example, the usual method of calculating the velocity is reduced to splitting the mean quantity $\overline{\rho u^2}$ in the quantity in the quantity $\bar{\rho}$, determined by dividing p by $R\bar{T}$ and subsequent evolution. The quantity ρu^2 is usually measured by means of a Pitot tube which yields something as a mean of $\overline{\rho u^2}$, $\bar{\rho}\overline{u^2}$ and $\bar{\rho}\bar{u}^2$. The difference between these values and the others in the calculations is, as a rule, not taken into account when processing the experimental data and comparing them with the results of calculations. In most cases, it is concealed by the spread of the experimental points and practically cannot be taken into account. In individual cases, in zones of steep temperature gradients (e.g., at the flame front in homogenous combustion of gas, etc.) this inaccuracy may result in noticeable not only quantitative but also qualitative distortions of the calculated velocity distribution.

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4, 5, 8, 11, 13, 20, 25, 26, 33, 41, 43, 44, 48, 50, 52, 55, 58, 73, 75, 77, 88, 114, 122, 136, 141, 157, 159, 165, 166, 169, 175, 181, 193, 194, 196, 200, 204, 212, 213, 215, 216, 222, 223, 224, 225, 229, 238, 239, 240, 242, 247, 250, 251, 252, 254, 255, 259, 260, 261, 263, 265, 266, 277, 278, 279, 280, 281, 282, 283, 284, 289, 293, 295, 305, 307, 311, 312, 314, 317, 320, 321, 323, 325.

Manu-script Page No.	[Footnotes]
217	We are concerned with an evolved turbulent flow. With relatively small values of Re its influence may be noticeable near the nozzle; it may affect the initial thickness of the jet, the position of the "pole," etc.
222	In particular, in the book [11] detailed data of G.S. Antonova [20] are given which refer to measurements of the pulsations of velocity, temperature and the correlation coefficients in a nonisothermal jet.
228	For the electric resistance of a wire heated by current, the case illustrated in Fig. 9.11a corresponds to the drop of resistance caused by cooling at high velocities and the increase of it because of heating by a high-temperature gas, and that of Fig. 9.11b corresponds to common heating or cooling of a wire by jets streaming against it. The sign of the correlation of the actions of u' and T' on the wire is therefore opposite to the sign of the correlation of u' and T' in the stream.

Chapter 10

TURBULENT MOMENTUM AND HEAT TRANSFER IN JETS OF INCOMPRESSIBLE FLUID

10.1 ON THE COEFFICIENTS OF TURBULENT EXCHANGE

A calculation of the flow pattern, the heat transfer and the like in the turbulent boundary layer, for example, in the case of a plane or axisymmetric nontwisted jet is reduced to an integration of equations in terms of averaged quantities, of the form

$$\left.\begin{aligned} &\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{1}{y^k} \frac{\partial}{\partial y}(y^k \tau_T), \\ &\frac{\partial}{\partial x}(y^k u) + \frac{\partial}{\partial y}(y^k v) = 0, \\ &\rho c_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}\right) = \frac{1}{y^k} \frac{\partial}{\partial y}(y^k q_T). \end{aligned}\right\} \qquad (10.1)$$

In these equations the tangential turbulent frictional stress τ_T and the turbulent heat flux q_T are, generally speaking, unknown functions of coordinates, averaged velocities and their derivatives. The expressions for τ_T and q_T written formally according to Bussineskiy [212] in the form of

$$\tau_T = \mu_T \frac{\partial u}{\partial y}, \quad q_T = -\lambda_T \frac{\partial T}{\partial y} \qquad (10.2)$$

replace the one unknowns by others as the question of the expressions for the unknown quantities μ_T and λ_T remains still open. In the general case, they are variable in the field of flow, complex functions of the coordinates and the velocity variation.

The assumption of the constancy of these quantities which permitted their treatment in analogy to the physical constants μ and λ, the molecular viscosity and the thermal conductivity, which, at first sight,

is quite natural, proved inapplicable in the general plan. Note however (as this was explained much later) that for certain particular cases (axisymmetric turbulent source jet of incompressible fluid) this assumption results in a satisfactory agreement with experiment wherefrom the empirical constants μ_T and λ_T are derived.

The solution of the system of equations (10.1) requires another way since a transition to the Reynolds equations again means replacing the one unknowns (τ_T and q_T) by others (the quantities $\overline{u'v'}$, $\overline{u'T'}$ etc.). It was therefore only for several decades that the semiempirical methods of the theory of turbulence, based on the ideas of L. Prandtl et al., resulted in a practical application of the turbulent boundary layer equations for the solution of the problem on the expansion of jets of incompressible fluids.

Since L. Prandtl's semiempirical theory of momentum transfer and J. Taylor's semiempirical theory of vorticity transfer are considered in detail in a great number of monographs and manuscripts on the mechanics of fluids and gases [11, 25, 130, 135, 151, 174, 175, 212 etc.] we restrict ourselves to a brief information dealing chiefly with the mathematical side of the problem.

The well-known formula L. Prandtl suggested in 1925 [278] for an incompressible fluid

$$\tau_T = \rho l^2 \left(\frac{\partial u}{\partial y}\right)^2; \quad l = cx, \qquad (10.3)$$

was used by W. Tollmien [317] when solving problems on the boundary of jet and plane and axisymmetric sources by the method of the turbulent boundary layer of finite thickness.

For the thermal problem an expression analogous to Eq. (10.3) for the heat flux will read as follows:

$$q_T = \rho c_p l l_q \frac{\partial u}{\partial y} \frac{\partial T}{\partial y}. \qquad (10.4)$$

Maintaining formally the expression of Eq. (10.2) we determine the coefficients of turbulent transfer in Eqs. (10.2) and (10.3) in the following way:

$$\nu_T = \frac{\mu_T}{\rho} = l^2 \frac{\partial u}{\partial y}, \quad a_T = \frac{\lambda_T}{\rho c_p} = l l_q \frac{\partial u}{\partial y} \qquad (10.5)$$

(in Prandtl's method of momentum transfer the "mixing lengths" for the velocity l and the temperature l_q are considered identical).

Formulas (10.3) and (10.4) are the basis of the theory of the turbulent boundary layer of finite thickness.

This method was developed in detail and applied in the papers by G.N. Abramovich [4] where the calculations are based on J. Taylor's method [4, 11] instead of that by L. Prandtl. It is well-known that the former yields the same result as Prandtl's theory for the solution of the dynamic problem. For the thermal problem Taylor's procedure has the advantage of different values for the velocity and temperature mixing lengths. Formally this difference corresponds to a value of $Pr_T = 0.5$ of the turbulent Prandtl number. This value is in qualitative agreement with the results of experiments whereas Prandtl's procedure resulted in the physically wrong conclusion of the similarity of the temperature and velocity distributions in the jet, the wake, etc.

At the same time, as mentioned long ago [174, 58, etc.] a calculation according to Taylor's method also yields results inadmissible from the physical point of view. As an example may serve the rectilinear temperature distribution (with discontinuities of the heat flux values at the boundaries) in the problem of the turbulent boundary of a incompressible fluid jet.

As regards the quantitative side of the problem, the equality $Pr_T = 0.5$ is also in disagreement with many experiments on nonisothermal turbulent jets of incompressible fluid. The failures of Taylor's

method become particularly marked when one tries to generalize them to turbulent jets of compressible gas. The remark is general in a well-known way and refers to the application of the method of the layer of finite thickness to the thermal problem. In fact neither the assumption on equal thicknesses of the dynamic and thermal boundary layers nor artificial ways of introducing different thicknesses succeed, as a rule, in obtaining results for the temperature distribution which are fully satisfactory from the physical point of view.

These disadvantages become particularly obvious in the calculation of compressible gas jets.

Much more convenient for the calculation is the so-called second method by L. Prandtl [280] (applied independently for one of the concrete flows by B.Ya. Trubchikov [196] in 1938). This method is the basis of a calculation of turbulent jets by the method of the asymptotic layer. In order to pass over to this scheme we replace in Prandtl's formula (10.5) a derivative by a ratio of finite differences, i.e., we assume that

$$\frac{\partial u}{\partial y} \sim \frac{u_{max} - u_{min}}{b}, \quad b \sim l \sim x,$$

and finally,

$$\nu_T = b(u_{max} - u_{min}), \quad a_T = b_q(u_{max} - u_{min}). \qquad (10.6)$$

The first of these formulas was applied by H. Görtler [250] in order to solve a series of dynamic problems (parallel flows, plane jet, etc.).

It is quite natural that any form of the expressions for the coefficients ν_T and a_T is expedient insofar as the form of the dependence of the transfer coefficients on the coordinates and the averaged velocity is "guessed" correctly for concrete flow problems.

In his time it was Prandtl's great merit that, assuming in the

formula $\tau = -\rho\overline{u'v'}$ for the velocity pulsation $u' \sim v' \sim l\frac{\partial u}{\partial y}$, he replaced the quantity $\overline{u'v'}$ which is difficult to guess by the clear notion of the mixing length l. In simple cases of a developed turbulent flow on a plate, in a jet or the wake, we may conclude from dimensionality considerations that $l \sim b$, where b is the characteristic thickness of the boundary layer. In self-similar flows along an infinite plate $b = c_1 y$, in a source jet (and in the problem of the jet boundary) $b = c_2 x$. The constants c_1 and c_2 must be determined by experiments.

The disadvantages of the procedure based on Eq. (10.6) (as also of the first Prandtl method) are connected with the assumption of the equality of the effective thicknesses of the dynamic and thermal layers, which results in a similarity of the velocity and temperature distribu tions. This is easy to avoid if in Eqs. (10.6) for ν_T and a_T we take different values of the characteristic thicknesses (or different values of the factors of proportionali ' in the formulas of the form $b = kx$, $b_q = k_q x$). This modification of the formulas is in fact not a logical consequence of the transformation of Prandtl's first formula to the second one. These considerations, however, cannot be considered as essential since proportionality formulas of the form $\nu_T \sim a_T \sim bU$ are obtained for the self-similar flow as the result of simple consideration of dimensionality. But as to the values of the factors of proportionality in these formulas, there is no reason for assuming them equal.

10.2. CONSIDERATIONS OF DIMENSIONALITY

Let us assume, for example, that in the case of a turbulent sourc jet in the self-similar zone of the flow the coefficients $\nu_T = \nu_T(x)$ and $a_T = a_T(x)$ depend only on the distance from the source. For a developed turbulent flow the dimension of the coefficient of kinematic viscosity can only be obtained in the form of a product of some characteristic length and a characteristic velocity. As the characteristic quantity

for the dimension we take the effective thickness of the jet $b = b(x)$, as the characteristic velocity we choose the velocity at the axis of the flow. Consequently,

$$\nu_T(x) = \text{const}\, b(x)\, u_m(x).$$

In analogy to this, the coefficient of turbulent thermal diffusivity, a_T, by virtue of the same dimension considerations, must be proportional to the coefficient of kinematic viscosity, ν_T, such that $a_T = \text{const}\; b(x)u_m(x)$.

Generally speaking, the expressions for ν_T and a_T contain different factors of proportionality, such that we could say that we are concerned with the transfer of quantities of vector or scalar character. Since the conception of the effective thickness of the asymptotic layer is based on convention, a difference in the factors of proportionality corresponds to different dimensions of the thermal and dynamic layers.

The problem whether the effective layer thicknesses are equal or different is, obviously, solved by way of comparison with the experiment. It has already been pointed out in the previous chapter that in all cases of free turbulent jets of incompressible fluid the thermal boundary layer is broader than the dynamic one; in other words, $a_T > \nu_T$.

Thus, for the mathematical scheme of the asymptotic layer we must assume in problems of source jets and the jet boundary that

$$\nu_T = bu_m; \quad a_T = b_q u_m. \qquad (10.7)$$

In this case, the turbulent Prandtl number $Pr_T = \frac{\nu_T}{a_T} = \frac{b}{b_q}$ will be a constant quantity in a first approximation.

The above considerations are based on a dimensionality analysis. It is therefore expedient to complete its application and establish the form of its dependence on the coordinate x of the effective layer thicknesses and the maximum velocity u_m (and also the characteristic value of the surplus temperature ΔT_m). Let us do this for three cases

of turbulent jet flows of incompressible fluid, namely a jet boundary, plane and axisymmetric source jets, under the supposition of universality of the distributions of u and ΔT in the jet cross sections.

As we are concerned with an evolved turbulent self-similar motion, the coefficient of molecular kinematic viscosity, ν, does not belong to the given quantities which do not comprise any quantity of the dimension of a length, with the exception of the coordinate x. The conditional value of the ordinate y_b corresponding to the effective thickness b of the layer must therefore be proportional to the coordinate x in all three problems. We can therefore write

$$b = Kx \quad \text{and} \quad b_q = K_q x; \frac{K}{K_q} = \mathrm{Pr}_T = \text{const.}$$

We determine the law of variation of the characteristic velocity $u_m = u_m(x)$ taking the integral condition of momentum conservation for source jets into account. For the jet boundary it can be given immediately: $u_m = u_0 = \text{const}$; we also have $\Delta T_m = \text{const}$. In this case the coefficient of kinematic viscosity $\nu_T \sim x$, and also $a_T \sim x$.

For a plane source jet $\frac{J_x}{\rho u_m^2} = \text{const}\, x \left(J_x = \int\limits_{-\infty}^{+\infty} \rho u^2\, dy = \text{const}\right)$, hence we obtain $u_m \sim \frac{1}{\sqrt{x}}$ and the coefficient of turbulent viscosity is $\nu_T \sim b u_m = K x u_m \sim \sqrt{x}$ and $a_T \sim \sqrt{x}$.

We find analogously that $\frac{Q}{\rho c_p u_m \Delta T_m} = \text{const}\, x \left(Q = \int\limits_{-\infty}^{+\infty} \rho c_p u \Delta T\, dy = \text{const}\right)$. Consequently, $\Delta T_m \sim u_m \sim \frac{1}{\sqrt{x}}$.

For an axisymmetric jet

$$\frac{J_x}{\rho u_m^2} = \text{const}\, x^2 \qquad \left(J_x = 2\pi \int\limits_0^{\infty} \rho u^2 y\, dy\right),$$

$$\frac{Q}{\rho c_p u_m \Delta T_m} = \text{const}\, x^2 \qquad \left(Q = 2\pi \int\limits_0^{\infty} \rho c_p u\, \Delta T y\, dy\right).$$

Correspondingly, $u_m \sim \Delta T_m \sim \frac{1}{x}$. Since $b \sim b_q \sim x$, the coefficients of turbulent kinematic viscosity and thermal diffusivity are constant for an axisymmetric jet:

$$\nu_T = bu_m = \text{const}, \quad a_T = b_q u_m = \text{const}.$$

The ratio of these constants again yields an empirical constant, namely the turbulent Prandtl number $Pr_T = \frac{\nu_T}{a_T}$.

From analogous considerations of the fact that, if the problem considered is well-defined, i.e., with the boundary conditions of all three cases, there does not exist a quantity that would permit the derivation of a characteristic dimension, it follows that the nondimensional argument for the universal velocity distribution will be a composite quantity: $\varphi = \text{const}\,\frac{y}{x}$. The conditions of self-similarity for the flows mentioned above are formulated in essentially the same way.

We see from the derivation that these conditions are independent of the assumptions on the dependence of the coefficients ν_T and a_T on the coordinate x alone. In the general case, we may put $\nu_T = f(\varphi) bu_m$, and the expressions for $u_m(x)$ given above remain still valid and the jet boundaries also remain rectilinear. For example, for Prandtl's first formula in the problem of the jet boundary, the plane or axisymmetric source jet, analogous considerations would have yielded correspondingly more general formulas of the form

$$\nu_T = f_1(\varphi)\,x, \quad \nu_T = f_2(\varphi)\sqrt{x}, \quad \nu_T = f_3(\varphi) \text{ with } \varphi = \text{const}\,\frac{y}{x}.$$

It must be borne in mind that all these conclusions only apply to self-similar jets since, for example, when we analyze the motion in a jet of finite dimensions, the boundary conditions must contain the nozzle diameter. In this case all conclusions on an unambiguous dependence of b, u_m, ν_T, etc. on the coordinate x would become senseless.

Analogous results based on the theory of dimensionality for self-similar jets have been given by H. Squire [300] who went still further when replacing the derivatives by finite differences. According to

H. Squire $\tau \sim \rho u_m^2 \sim \frac{1}{x^2}$ and $q_T \sim \frac{1}{x^2}$.

As regards the problem of the expansion of a turbulent jet in a parallel flow, it is only the mixing zone at the boundary of two parallel flows for which the above conclusions remain valid with a characteristic velocity value of $u_m = u_{01} - u_{02} = \text{const}$. For a source jet (plane or axisymmetric) which expands in the wake, the conditions of the problem will contain an additional dimensional quantity, namely the velocity of the wake; when it is taken into account, the initial conditions for, e.g., a plane jet will contain the characteristic length $L = \frac{J_x}{\rho u_\infty^2}$.

When discussing the results it must be pointed out that the introduction of the conception of effective coefficients of turbulent viscosity and thermal conductivity and the choice of simple mathematical formulas for the determination of these coefficients are justified for self-similar flows. In the general case of jets of finite dimensions and the like, semiempirical theories of the type of the mixing length or analogous mathematical procedures based on dimensionality considerations leave the problem of the values of the coefficients ν_T and a_T, their dependence on the coordinates and the velocity, etc. unsolved and do not yield a closed mathematical system.

A solution of the problems of the laminar boundary layer theory (e.g., the problem of the expansion of a jet of finite dimensions, parallel jets, etc.) is at present only connected with the overcoming of difficulties of mathematical nature. Unlike that in the theory of the turbulent boundary layer for the solution of complex problems on nonself-similar jet flows one needs information on the structure and the dependence on coordinates and velocity of the coefficients of turbulent exchange (or immediately of the turbulent friction and the heat flux), which is not available at present. Precisely therefore

for the noninvestigated turbulent jet flows a preliminary solution of the corresponding problems on the expansion of laminar jets results in the development of models qualitative in their way.

10.3. THE SELF-SIMILARITY TRANSFORMATION FOR TURBULENT JETS

Just as in the theory of laminar jets, we again use the self-similarity transformation formulas in the form [57]

$$\frac{u}{u_m} = \frac{F'(\varphi)}{\varphi^k}, \quad u_m = Ax^{\alpha}, \qquad \varphi = Byx^{\beta}.$$
$$\frac{\Delta T}{\Delta T_m} = \theta(\varphi), \quad \Delta T_m = \Gamma x^{\gamma},$$

In the previous section it has been shown that for all free turbulent jets with self-similar flow the constant $\beta = -1$. This important result also remains valid for the fan-type source jet, plane and fan-type semilimited jets and the like. This indicates that the effective boundary of the turbulent boundary layer (unlike the laminar one) in self-similar jet motion is always rectilinear. It is obvious that the same result can be obtained from differential equations describing the expansion of a jet of incompressible fluid. As always, the investigation of the differential equations of motion together with the integral conditions of conservation yields more than a simple analysis of dimensions. In the given case with the help of the differential equations and the integral conditions, it is possible not only to verify the values of the self-similarity constants α and β obtained above from dimensionality considerations, but also to determine the constants A and B. For turbulent jets, however, the latter are determined accurately except for a numerical factor whose value is obtained by experiment.

It is extraordinarily important – as will be shown below – that the self-similar differential equations for turbulent jet motions under definite conditions are identical with the analogous equations for laminar jets (for the same types of motion). The values of the constants of

self-similarity, α, β and γ and the constants A, B and Γ for the laminar and turbulent motions are here different.

If we assume, as already mentioned, the values of the constants α and γ equal to 0, 1/2, 1, respectively, for the jet boundary, the plane and the axisymmetric source jets, we obtain the ordinary differential equations given previously in Table 7.1 for laminar jets.

For a plane source jet, for example, the equations read

$$F''' + 2(FF')' = 0, \quad \theta'' + 2\mathrm{Pr}_T(F\theta)' = 0$$

(with the same boundary conditions as in the laminar jet). The values of the constants A, B and Γ entering the transformation formulas for the turbulent plane source jet will be equal to

$$A = \sqrt{\frac{3J_x}{8\rho\sqrt{K}}}, \quad B = \frac{1}{2\sqrt{K}}, \quad \Gamma = \frac{Q}{c_p}\sqrt{\frac{2}{3\rho J_x\sqrt{K}}}\left[\int_{-\infty}^{+\infty} F'\theta\, d\varphi\right]^{-1}.$$

These values are connected with the numerical coefficient K (encountered in the formula $b = Kx$) and the turbulent Prandtl number given by $\mathrm{Pr}_T = \frac{K}{Kq}$. The value of one of the constants (usually $a \equiv B^{-1} = 2\sqrt{K}$ in the case of the plane jet) and also that of the Prandtl number is chosen empirically, by way of comparing the experimental and the theoretical velocity and temperature distributions.

Under these conditions the problem can be solved. The universal distributions $F'(\varphi)$ and $\theta(\varphi)$ are obtained in the same way as in the case of laminar jets with corresponding boundary conditions.

Analogous results (the same ones as with laminar jets, i.e., the form of equations and solutions) also correspond to the problems of the axisymmetric source jet and the jet boundary. The same also holds true for the fan-type source jet.

As already proven in the theory of laminar jets, for self-similar flows, immediately from the transformation formulas of the form $\frac{u}{u_m} = F'(\varphi)$, $u_m = Ax^{\alpha}$, $\varphi = Byx^{\beta}$ the existence of an integral equation of the

form

$$\int_{(S)} u^{\sigma} ds = \text{const}, \quad \sigma = \frac{l\beta - k}{\alpha} \qquad (10.8)$$

results (and an analogous one for the temperature). Equation (10.8) applies likewise to laminar and turbulent self-similar jets. For free jets the exponent in (10.8) $\sigma = 2$ and the expression itslef coincides with the more general condition of momentum conservation (independently of the assumption of self-similarity). For turbulent jets $\beta = -1$. It therefore follows from Eq. (10.8) that the exponent of u is equal to

$\sigma = -1/\alpha$ in the case of a plane source jet,

$\sigma = -2/\alpha$ for axisymmetric and fan-type jets.

Hence it follows that with $\sigma = 2$ for free turbulent self-similar jets $\alpha = -1/2$ and $\alpha = -1$, respectively, for plane and axisymmetric (and fan-type) source jets.

More detailed data on free turbulent self-similar jets are given in Table 11.1. As regards the semilimited jets, for them (for both laminar and turbulent ones) $\sigma = 3/2$. With $\beta = -1$, for turbulent plane and axisymmetric (also fan-type) jets it follows from Eq. (10.8) that $\alpha = -2/3$ and $\alpha = -4/3$, respectively. It is, however, necessary to stress that the simple supposition of self-similarity of turbulent semilimited jets does not yield good agreement with the experiment. The d vergence is caused by the differences in structure of the free and near-to-the-wall turbulent boundary layers. The values of the constant α given here are therefore only approximate though they apply virtually always to the outer parts of semilimited jets. Owing to these considerations, the results for turbulent semilimited jets are not entered in Table 11.1 but considered separately.

10.4 ON THE TURBULENT PRANDTL NUMBER

One of the constants which are determined by experiement is the

ratio of the effective coefficients of turbulent exchange, $Pr_T = \frac{\nu_T}{a_T} = \frac{K}{K_q}$. This characteristic parameters, unlike the molecular Prandtl number $Pr = \nu/a$, as also the coefficients ν_T and a_T themselves, does not belong to the physical constants. This number characterizes the complex and actually hydrodynamic phenomenon of the interaction between the transfer of a vector quantity (momentum) and that of a scalar characteristic parameter (heat). As already mentioned, the supposition on the equality of ν_T and a_T (which corresponds to an assumed likeness of nature of the momentum and heat carriers in the turbulent exchange, the turbulent pulsations) which is a natural supposition only at first sight, is in contradiction with the experiement. For turbulent jets, the latter always shows that $\nu_T < a_T$. As regards the numerical value of Pr_T it is, when determined from experiments on the expansion of air jets [237, 238, etc.], very close to the physical Prandtl number Pr for air $(Pr \approx 0{,}72)$.

The similarity of the values of Pr and Pr_T for air sometimes gave rise to the opinion that this agreement were not accidental. Though this assumption is not based on any rational considerations, a final solution of the problem would require special experiments. Such experishould be carried out with a fluid medium whose physical Prandtl number differs essentially from unity.

This "decisive" experiment was carried out by Z.B. Sakipov [165, 166]. In experiments on the expansion of a slightly heated axisymmetric jet of viscous oil with a physical Prandtl number of $Pr \sim 10^3$ and in analogous experiments with mercury jets for which $Pr \sim 10^{-2}$ the distributions of velocity and surplus temperature were measured carefully.

The experimental results were evaluated in the form of the universal distribution functions $\frac{u}{u_m} = F(\varphi)$ and $\frac{\Delta T}{\Delta T_m} = 0(\varphi)$. The distributions obtained had a form which is usual for turbulent jets (in particular, a

value of the constant $a \equiv \frac{1}{B}$ in the formula $\varphi = \frac{y}{ax}$ of $a \approx 0{,}08$ for mercury and $a \approx 0{,}07$ for oil corresponded to it).

From the solutions for the self-similar section of the flow (source jet) in the theory of the asymptotic layer and also that of the layer of finite thickness, it follows that the universal temperature and velocity distributions in a jet cross section are mutually related by a formula (see Tables 7.1 and 11.1) of the form

$$\frac{\Delta T}{\Delta T_m} = \left(\frac{u}{u_m}\right)^{\mathrm{Pr}_T}.$$

This relation is suitable for the determination of the mean value of the turbulent Prandtl number Pr_T.

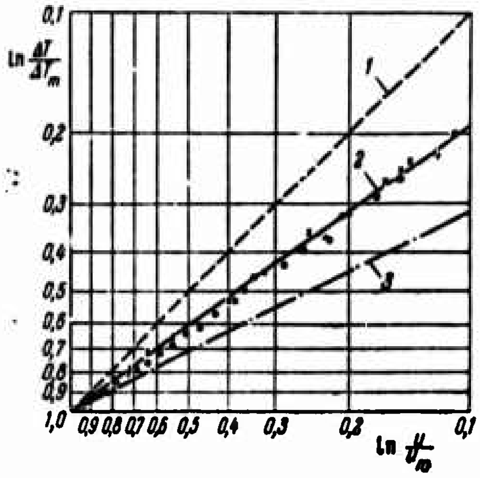

Fig. 10.1. To the determination of the turbulent Prandtl number (after data by Z.B. Sakipov [165, 166]). 1) $\mathrm{Pr}_T = 1.0$ (Prandtl method); 2) $\mathrm{Pr}_T = 0.75$ (experiment), xxx—oil, ooo—mercury; 3) $\mathrm{Pr}_T = 0.5$ (Taylor method).

Figure 10.1 shows in a semilogarithmical scale $\frac{\Delta T}{\Delta T_m}$ as a function of u/u_m obtained in experiments by Z.B. Sakipov with transformer oil and mercury. As seen from the figure, the experimental data fit quite well the straight line with the angular coefficient $\mathrm{Pr}_T \approx 0{,}72 \div 0{,}75$.

Thus, in these experiments, in the broadest range of values of

the physical Prandtl number direct results were obtained which prove that in an evolved free turbulent flow the turbulent Prandtl number is independent of the physical Prandtl number.

Somewhat earlier W. Forstall and E. Gaylord [242] studied experimentally an axisymmetric water jet expanding in a salt solution. In these experiments, the velocity and concentration distributions were measured in the cross sections of the self-similar part of the jet. The experimental results, which were evaluated in the form of a dependence of the nondimensional excessive concentration on the relative velocity, correspond to a value of the turbulent diffusional Prandtl number $Pr_{T,dif} \approx 0.7-0.8$, whereas the physical diffusional Prandtl number is $Pr_{dif} \approx 10^3$. These results also agree with other more accurate data given by various authors on the intensity correlation of processes of turbulent transfer (diffusion, heat exchange, viscosity) in gases, among them also in the free atmosphere [298, 169].

All these results, in particular the direct experiments by Z.B. Sakopov, show that in the broadest range of physical properties of a fluid the so-called turbulent Prandtl number in a free turbulent motion is always smaller than unity. Its numerical value lies obviously between 0.7-0.8 such that a value of $Pr_T \approx Pr_{T.\text{диф}} \approx 0{,}75$. can be taken as the mean. The agreement of this value with the physical Prandtl number of air is quite accidental. More than that, as will be shown later, the value of $Pr_T \approx 0{,}75$ also applied approximately to turbulent jets of compressible gas. This value is a mean value for jets. Detailed data on the distribution of the Pr_T value over the jet cross section (not only for different values of Prandtl's number Pr but also for air) are not available at present.

LITERATURE REFERENCES

4, 11, 12, 25, 41, 47, 48, 49, 57, 58, 69, 114, 115, 130, 134, 151, 157, 165, 166, 169, 174, 175, 193, 196, 206, 212, 213, 214, 237, 238, 239, 242, 250, 251, 252, 261, 277, 278, 279, 280, 281, 282, 283, 284, 285, 289, 298, 300, 310, 317, 321, 322, 325.

Chapter 11

SELF-SIMILAR TURBULENT JETS OF INCOMPRESSIBLE FLUID

11.1. ON THE METHODS OF CALCULATION

The theory of turbulent jets and its numerous applications in engineering are based on investigations of self-similar jet flows of incompressible fluids, which were first carried out by means of the method of the finite layer, using Prandtl's formula of the mixing length for the three simplest problems, namely the problem of the edge of jet, the plane source jet and the round source jet (see [317]).

At present we possess various methods in order to solve these problems semiempirically. Among them the following methods, which have already been mentioned previously, are worth emphasizing: the methods of the finite and asymptotic layers, the "constructive" methods (upon which the solution of given velocity distributions, etc., are based), G. Reichardt's phenomenological theory, and others. A sufficiently comprehensive idea on these methods and their applications to concrete problems can be obtained from the original papers or review articles mentioned in the reference list.

A detailed review on the various methods of calculating self-similar turbulent jets may be found, for example, in the monograph by G.N. Abramovich [11], in the book by Wai Shi-i [25], in manuscripts on the boundary layer [174, 212], etc.

This fact and the subject chosen for the present book render it

superfluous to enter into details of these methods. Let us therefore only consider the results of solutions of various problems obtained within the framework of the theory of the asymptotic boundary layer as they permit the application of the final solutions of the theory of the laminar self-similar jets in the calculation of turbulent jets. As regards the other methods, we restrict ourselves to some remarks of general nature.

The essence of these remarks results in the following.

In spite of the apparent differences in the premisses and the system of mathematical operations, the various methods of calculating the self-similar turbulent jets are essentially general in the sense that they unite in themselves certain data which are derived immediately from the experiment (represented by the empirical constants, and the one or other general aspects of the mechanics of viscous fluids. The latter are used in the form of differential equations of motion, integral relations or, more approximately, in the form of a "guessed" universal distribution. Similarity considerations are of decisive importance in this respect as they verify the self-similarity of the flows.

In the case of self-similar flows the various mathematical methods yield results which, in the end, resemble one another and which, within well-known limits of accuracy (mostly within the limits of spread of the experimental data) agree with the experimental results if the experimental constants are chosen correspondingly. These constants do not play the part of universal constants, their values may vary, generally speaking, within wide limits though, under the so-called "ordinary" conditions, they vary not very strongly. The latter fact enabled G.N. Abramovich [11] to introduce – with well-known success – a single constant for an approximate calculation of various self-similar jet flows.

It is obvious that the numerical values of the experimental constants which are introduced, for example, in the form of a "structural constant" of the jet, a, in the formula for the nondimensional variable $\varphi = \frac{y}{ax}$ are different in different mathematical procedures. In order to compare them, it is expedient to superpose universal distributions of the form of $\frac{u}{u_m} = f(\varphi)$ such that two values of the relative velocity coincide: $u/u_m = 1$ and, usually, $u/u_m = 1/2$. As a rule in a considerable part of the distribution (e.g., within the limits of $0{,}25 < \frac{u}{u_m} < 1$) the different solutions agree with one another and with the experiment within the limits of accuracy of the latter. In the lateral parts of the distributions the deviations are more considerable. In particular, the method of the asymptotic layer, compared with the experiment, yields a somewhat delayed drop of velocity, temperature, and the like, in the jet cross sections. But even this deviation remains in most cases within the limits of experimental accuracy (generally speaking, the jet boundary layer which is thin near the boundaries will there influence the alternating character of the flow, as already mentioned.)

These considerations show that we have no sufficient reasons speaking in favor of a unique choice of some semiempirical mathematical procedure and the rejection of all others. In individual concrete cases the one or another way of solution may prove somewhat simpler or more convenient.

Considering what has been said in the preceding sections, the authors are of the opinion that the method of the asymptotic layer, which links the self-similar solutions for turbulent jets with the physically strict theory of the laminar jets, possesses a series of advantages, as it also describes correctly the relationships between the thermal and dynamic layers (when the empirical constant is chosen as $Pr_T \approx 0{,}7 \div 0{,}8$) At last, and this plays a well-known part, the final

formulas in the theory of the asymptotic layer can often be represented in the form of closed analytical expressions (and thus make it possible to avoid the approximate numerical constructions which are so often encountered in the theory of the finite layer, and their "joints" within the framework of one and the same distribution, etc.).

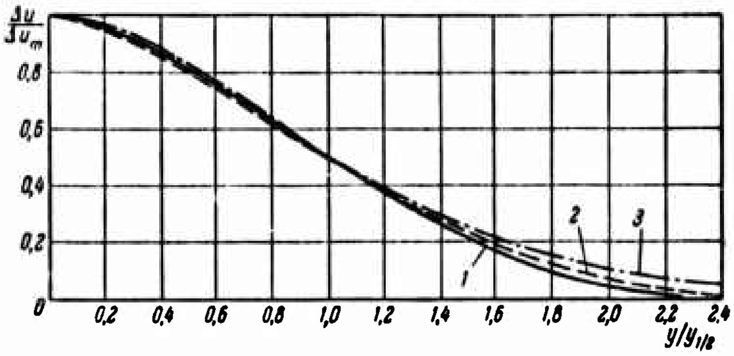

Fig. 11.1. Comparison of calculated distributions of velocity difference in parallel flows. 1) After the finite-layer method $(\Delta u/\Delta u_m = [1 - (y/y_{1/2})^{3/2}]^2$ Schlichting's distribution [11]; 2) the same(W. Tollmien [317]); 3) according to the asymptotic-layer method (H. Görtler [250]).

All this, of course, does not deny the value of other methods and the possibility of obtaining with them valuable and original results as such are described particularly in the papers [11, 25, 75, 130, etc.].

In order to illustrate this, we shall consider the individual results of solutions of one and the same problem by different methods and compare them with the experiment.

In Fig. 11, which has been taken from a monograph by G.N. Abramovich [11], the relative distributions of the surplus velocity in the cross section of a parallel flow $\frac{\Delta u}{\Delta u_m} = f(\xi)$, where $\xi = \frac{y}{y_{1/2}}$, are compared. Two of them were obtained by the finite-layer method and the third one by the method of the asymptotic layer.

A comparison of the calculated velocity distribution curves for the jet boundary obtained according to the methods of W. Tollmien and

H. Görtler with H. Reichardt's experiments (and other experimental data) shows satisfactory agreement in both cases with appropriate choice of the empirical constant. Thrre is virtually no difference between the results of the methods. Analogous data could be given in a great number; they can be found in the papers [11, 25, 174] etc.

As already mentioned, we shall use the method of the asymptotic layer for the further calculations. With this method, the equations for the self-similar jets and the boundary conditions, as also their solutions in nondimensional coordinates are the same as in the case of the laminar self-similar jets. Only the value of the constant of self-similarity and the constants entering the transformation formulas will be different.

For convenience, we use all these data for free turbulent jets as compiled in Table 11.1, which has been compiled analogously as Table 7.1 for the laminar jets. It also contains data on the dynamic and thermal problems for source jets and the mixing zone of parallel and antiparallel uniform flows which are dealt with in detail in one of the following sections.

As regards the turbulent semilimited jets, the next chapter will be devoted to it. Finally, it will be expedient to consider the expansion of turbulent jets of finite dimensions, owing to the generality of the method, for both the incompressible fluid and the compressible gas at the same time.

11.2. RESULTS OF SOLUTIONS IN THE CASE OF FREE JETS

In this section we shall give loose data for the fundamental problems of the expansion of turbulent self-similar jets (free plane jet, fan-type jet, and axisymmetric twisted and nontwisted source jets and the mixing zones at the boundary of two plane-parallel flows). All these solutions have the self-similarity transformation formulas in

common which read

$$\frac{u}{u_m} = \frac{F'(\varphi)}{\varphi^k}, \quad \frac{v}{au_m} = f(\varphi), \quad \frac{w}{au_m} = \Phi(\varphi),$$
$$\frac{\Delta p}{\Delta p_m} = P(\varphi), \quad \frac{\Delta T}{\Delta T_m} = \theta(\varphi),$$

where

$$u_m = Ax^\alpha, \quad \Delta p_m = Dx^\delta, \quad \Delta T_m = \Gamma x^\gamma, \quad \varphi = \frac{y}{ax}.$$

(For the sake of convenient comparison with the results of other papers we shall write the independent variable φ in the form $\varphi = Byx^\beta \equiv \frac{y}{ax}$, where $a \equiv \frac{1}{B}$ and $\beta = -1$.)

The results of these solutions are compiled in the general Table 11.1 where, for convenience of use, the fundamental data of Table 7.1 for laminar jets and the expressions defining the dependence on the coordinate x of the integral characteristics (flow rate G of fluid through jet cross section, flux Q of heat content, kinetic energy flux E, and the effective coefficients of turbulent transfer, ν_T and a_T) have been repeated. Though the value of the turbulent Prandtl number varies but little in the problem considered, the symbol Pr_T (and not its numerical value which, as already mentioned, is approximately equal to $Pr_T \approx 0.75$) has been maintained.

The use of the formulas given in Table 11.1 in calculations makes it necessary to determine beforehand the value of the empirical constant a by means of an experiment. Recall that for real jets the self-similar flows, which are responsible for the concept of the source jet, are only established at a considerable relative distance from the nozzle. The choice of a concrete value of x/d (d being the nozzle diameter) beyond which the flow can be considered as self-similar depends on the degree of accuracy required.

Since the general condition of self-similarity of a jet flow is represented by an inequality of the form $\frac{x}{d} \gg 1$, the flow, strictly

speaking, can be considered self-similar at distances of the order of several ten nozzle diameters; in practice, with lower accuracy, we can reduce the limiting value of x/d to $\sim 15-20$, and sometimes even to a few diameters.

In the case where the source jet distribution is used for an approximate description of the real flow at relatively short distances from the nozzle it becomes necessary, apart from the constant a, also to determine the position of the equivalent (with respect to the momentum flux) source. The latter (the so-called "pole" of the jet according to G.N. Abramovich's terminology [4]), generally speaking, lies at a certain distance x_0 away from the orifice of the nozzle of finite diameter (behind or in front of the outlet cross section). NOte that when we speak of the equivalent source jet, we understand the flow from a source with the same total momentum J_x as the flow of finite dimension,

$$J_x = \int_s \rho u^2 ds = \rho u_0^2 s_0,$$

where u_0 is the mean value (with respect to the momentum) of the velocity in the nozzle's outlet cross section of the area s_0. In this case, the value of the coordinate x in the formulas of Table 11.1 must be replaced by the sum $\bar{x} + x_0$, where $\bar{x}$ is the distance from the nozzle.

TABLE 11.1

Free Self-Similar Turbulent Jets of Incompressible Fluid

Scheme of Table

1) Form of flow;
2) differential equations;
3) boundary conditions;
4) transformation formulas;
5) constants of self-similarity;
6) constants of the transformation formulas;
7) integral conditions of conservation.
8) self-similar equations;
9) the coordinate φ_* corresponding to the value of $u/u_m = 1/2$.
10) boundary conditions;
11) solutions;
12) integral (fluid flow rate, kinetic energy flux) and local (coefficients of turbulent exchange, v_T and a_T) characteristics.

Notes:

In the lines 2, 3, etc. I refers to the thermal problem and II to the dynamic one.

In the lines 3, 4, etc. IIa refers to symmetrical boundary conditions for the temperature, IIb to asymmetric boundary conditions for the temperature.

1		A Плоская струя-источник		
2	I	$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\nu_T\frac{\partial u}{\partial y}\right);\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,\ (\nu_T = KAx^{\alpha-\beta})$		
	II	$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left(a_T\frac{\partial T}{\partial y}\right)\ (a_T = K_q Ax^{\alpha-\beta})$		
3	I	$v(x,\ 0) = 0,\ \left.\frac{\partial u}{\partial y}\right\|_{y=0} = 0;\ u(x,\ \pm\infty) = 0$		
	IIa	$\left.\frac{\partial T}{\partial y}\right\|_{y=0} = 0,\ T(x,\ \pm\infty) = T_\infty$	IIб	$T(x,\ +\infty) = T_1,$ $T(x,\ -\infty) = T_2$

TABLE 11.1 continued

1		A Плоская струя-источник		
4	I	$\frac{u}{u_m} = F'(\varphi),\; u_m = Ax^{\alpha},\; \varphi = Byx^{\beta}\; \left(B \equiv \frac{1}{a}\right)$		
	IIa	$\frac{T - T_\infty}{T_m - T_\infty} = \theta(\varphi);\; T_m - T_\infty = \Gamma x^{\gamma}$	IIб	$\frac{T - T_2}{T_1 - T_2} = \theta(\varphi) \quad (\Gamma = T_1 - T_2)$
5	I	$\alpha = -\frac{1}{2},\; \beta = -1$		
	IIa	$\gamma = -\frac{1}{2}$	IIб	$(\gamma = 0)$
6	I	$A = \sqrt{\frac{3J_x}{8\rho\sqrt{K}}};\; B = \frac{1}{2\sqrt{K}}$		
	IIa	$\Gamma = \frac{Q}{c_p}\sqrt{\frac{2}{3\rho J_x\sqrt{K}}}\left[\int\limits_{-\infty}^{+\infty} F'\theta\, d\varphi\right]^{-1}$	IIб	$\Gamma = T_1 - T_2$
7	I	$J_x = \int\limits_{-\infty}^{+\infty} \rho u^2\, dy = \mathrm{const}$		
	IIa	$Q = \int\limits_{-\infty}^{+\infty} \rho c_p u (T - T_\infty)\, dy = \mathrm{const}$	IIб	—
8	I	$F''' + 2(FF')' = 0$		
	IIa	$\theta'' + 2Pr_T (F\theta)' = 0$	IIб	$\theta'' + 2Pr_T F\theta' = 0$
9	I	$\varphi_{1/2} = 0{,}88$		
10	I	$F(0) = 0,\; F'(0) = 1,\; F'(\pm\infty) = 0$		
	IIa	$\theta(0) = 1;\; \theta(\pm\infty) = 0$	IIб	$\theta(+\infty) = 1,\; \theta(-\infty) = 0$
11	I	$F = \mathrm{th}\,\varphi,\; F' = 1 - \mathrm{th}^2\varphi$		
	IIa	$\theta(\varphi) = (F')^{Pr_T} = (1 - \mathrm{th}^2\varphi)^{Pr_T}$	IIб	$\theta(\varphi) = \left[\int\limits_{-\infty}^{\varphi} (\mathrm{ch}\,\varphi)^{-2Pr_T} d\varphi\right]\left[\int\limits_{-\infty}^{+\infty} (\mathrm{ch}\,\varphi)^{-2Pr_T} d\varphi\right]^{-1}$
12	I	$G \sim x^{1/2},\; E \sim x^{-1/2};\; \nu_T \sim x^{1/2}$		
	IIa	$a_T \sim x^{1/2}$	IIб	$Q \sim x^{1/2},\; a_T \sim x^{1/2}$

1		C Спутные или встречные потоки
2	I	$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\nu_T \frac{\partial u}{\partial y}\right);\; \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad [\nu_T = K(u_1 - u_2)x^{\alpha-\beta}]$
	II	$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left(a_T \frac{\partial T}{\partial y}\right) \quad [a_T = K_q(u_1 - u_2)x^{\alpha-\beta}]$

TABLE 11.1 continued

1		C Спутные или встречные потоки		
3	I	$u(x, +\infty) = u_1, \left.\frac{\partial u}{\partial y}\right\|_{y=\pm\infty} = 0;\ u(x, -\infty) = u_2$		
	IIa	—	IIб	$T(x, +\infty) = T_1,\ T(x, -\infty) = T_2$
4	I	$\frac{u}{u_m} = F'(\varphi),\quad u_m = u_1 = \text{const},\quad \varphi = Byx^{\beta}\left(B \equiv \frac{1}{a}\right)$		
	IIa	—	IIб	$\frac{T - T_2}{T_1 - T_2} = \theta(\varphi)$
5	I	$\alpha = 0,\ \beta = -1$		
	IIa	—	IIб	$\gamma = 0$
6	I	$A = u_1,\ B = \frac{1}{a} = \frac{1}{\sqrt{2K(1-m)}}\quad \left(m = \frac{u_2}{u_1}\right)$		
	IIa	—	IIб	$\Gamma = T_1 - T_2$
7	I	—		
	IIa	—	IIб	—
8	I	$F''' + 2FF'' = 0$		
	IIa	—	IIб	$\theta'' + 2\mathrm{Pr}_T F\theta' = 0$
9	I	$\varphi_{1/2} \approx -0{,}33$ (при $m = 0$)		
10	I	$F'(+\infty) = 1,\ F''(\pm\infty) = 0;\ F'(-\infty) = m$		
	IIa	—	IIб	$\theta(+\infty) = 1,\ \theta(-\infty) = 0$
11	I	$F'(\varphi) = \frac{u}{u_1} = 1 + \frac{1}{2}(m - 1)[1 - \mathrm{erf}(\varphi + \varphi_0)].\quad (\varphi_0 \approx 0{,}33)$		
	IIa	—	IIб	$\theta = \frac{1}{2}[1 + \mathrm{erf}(\varphi + \varphi_0)\sqrt{\mathrm{Pr}_T}]$
12	I	$\nu_T \sim x$		
	IIa	—	IIб	$a_T \sim x$

1		D Осесимметричная струя-источник		
2	I	$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\nu_T}{y}\frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial y}\right);\ \frac{\rho w^2}{y} = \frac{\partial p}{\partial y};\ u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + \frac{vw}{y} = \nu_T\left\{\frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial w}{\partial y}\right) - \frac{w}{y^2}\right\};\ \frac{\partial}{\partial x}(yu) + \frac{\partial}{\partial y}(yv) = 0.$		
	II	$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = a_T\ \frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial T}{\partial y}\right)$		
3	I	$v(x, 0) = w(x, 0) = \left.\frac{\partial u}{\partial y}\right\|_{y=0} = 0;\ u(x, \infty) = w(x, \infty) = 0;\ p(x, \infty) = p_\infty$		
	IIa	$\left.\frac{\partial T}{\partial y}\right\|_{y=0} = 0;\ T(x, \infty) = T_\infty$	IIб	—

TABLE 11.1 continued

1		D Оссимметричная струя-источник		
4	I	$\frac{u}{u_m} = \frac{F'(\varphi)}{\varphi}$, $\frac{w}{w_m} = \Phi(\varphi)$, $\frac{p - p_\infty}{p_m - p_\infty} = P(\varphi)$, $u_m = Ax^\alpha$, $w_m = Cx^\varepsilon$, $p_m - p_\infty = \rho D x^\delta$; $\varphi = Byx^\beta$		
	IIa	$\frac{T - T_\infty}{T_m - T_\infty} = \theta(\varphi)$; $T_m - T_\infty = \Gamma x^\gamma$	IIб	—
5	I	$\alpha = -1,\ \beta = -1,\ \varepsilon = -2,\ \delta = -4$		
	IIa	$\gamma = -1$	IIб	—
6	I	$A = \sqrt{\frac{3J_x}{8\pi\rho K}}$; $B = \frac{1}{a} = \frac{1}{\sqrt{K}}$; $C = \frac{3M_x}{32\pi\rho K}\sqrt{\frac{8\pi\rho}{3J_x}}$; $D = \frac{M_x^2}{32\pi\rho K^2 J_x}$ $(K = a^2)$		
	IIa	$\Gamma = \left(\mathrm{Pr}_T + \frac{1}{2}\right)\frac{Q}{c_p} \cdot \frac{1}{\sqrt{6\pi\rho K J_x}}$	IIб	—
7	I	$J_x = 2\pi\int_0^\infty \rho u^2 y\, dy = \mathrm{const}$; $M_x = 2\pi\int_0^\infty \rho u w y^2\, dy = \mathrm{const}$		
	IIa	$Q = 2\pi\int_0^\infty \rho c_p u (T - T_\infty)\, y\, dy = \mathrm{const}$	IIб	—
8	I	$\left(F'' - \frac{F'}{\varphi}\right) + \left(\frac{FF'}{\varphi}\right)' = 0$; $P' = \frac{\Phi^2}{\varphi}$; $\Phi'' + \frac{1 + F}{\varphi}\Phi' + \frac{\varphi F' + F - 1}{\varphi^2}\Phi = 0$		
	IIa	$(\varphi\theta') + \mathrm{Pr}_T (F\theta)' = 0$	IIб	—
9	I	$\varphi_{1/2} = 1{,}82$		
10	I	$\left.\frac{F'}{\varphi}\right\vert_{\varphi=0} = 1$; $\left.\frac{F}{\varphi}\right\vert_{\varphi=0} = 0$; $\Phi(0) = 0$; $\left.\frac{F'}{\varphi}\right\vert_{\varphi=\infty} = 0$; $\Phi(\infty) = P(\infty) = 0$		
	IIa	$\theta'(0) = 0$; $\theta(\infty) = 0$	IIб	—
11	I	$F(\varphi) = \frac{\frac{1}{2}\varphi^2}{1 + \frac{1}{8}\varphi^2}$; $F'(\varphi) = \frac{\varphi}{\left(1 + \frac{1}{8}\varphi^2\right)^2}$; $\Phi(\varphi) = \frac{\varphi}{\left(1 + \frac{1}{8}\varphi^2\right)^2}$; $P(\varphi) = \frac{1}{\left(1 + \frac{1}{8}\varphi^2\right)^3}$		
	IIa	$\theta(\varphi) = \left(\frac{F'}{\varphi}\right)^{\mathrm{Pr}_T} = \left(1 + \frac{1}{8}\varphi^2\right)^{-2\mathrm{Pr}_T}$	IIб	—
12	I	$G \sim x$, $E \sim x^{-1}$; $\nu_T = KA = \mathrm{const}$		
	IIa	$a_T = K_q A = \mathrm{const}$; $\mathrm{Pr}_T = \frac{K}{K_q}$	IIб	—

1		E Веерная струя-источник
2	I	$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\nu_T \frac{\partial u}{\partial y}\right)$; $u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + \frac{uw}{x} = \frac{\partial}{\partial y}\left(\nu_T \frac{\partial w}{\partial y}\right)$; $\frac{\partial}{\partial x}(xu) + \frac{\partial}{\partial y}(xv) = 0$
	II.	$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left(a_T \frac{\partial T}{\partial y}\right)$

TABLE 11.1 continued

1		E Веерная струя-источник		
3	I	$v(x, 0) = 0, \left.\frac{\partial u}{\partial y}\right\vert_{y=0} = \left.\frac{\partial w}{\partial y}\right\vert_{y=0} = 0;\ u(x, \pm\infty) = w(x, \pm\infty) = 0$		
	IIa	$\left.\frac{\partial T}{\partial y}\right\vert_{y=0} = 0,\ T(x, \pm\infty) = T_\infty$	IIб	$T(x, +\infty) = T_1,$ $T(x, -\infty) = T_2,$
4	I	$\frac{u}{u_m} = F'(\varphi),\ \frac{w}{w_m} = \Phi(\varphi),\ u_m = Ax^\alpha,\ w_m = Cx^\varepsilon;\ \varphi = Byx^\beta\ \left(B = \frac{1}{a}\right)$		
	IIa	$\frac{T - T_\infty}{T_m - T_\infty} = \theta(\varphi);\ T_m - T_\infty = \Gamma x^\gamma$	IIб	$\frac{T - T_2}{T_1 - T_2} = \theta(\varphi)$
5	I	$\alpha = -1,\ \beta = -1,\ \varepsilon = -2$		
	IIa	$\gamma = -1$	IIб	$\gamma = 0$
6	I	$A = \sqrt{\frac{3J_x}{8\pi\rho\sqrt{2K}}};\ B = \frac{1}{\sqrt{2K}};\ C = M_y\sqrt{\frac{3}{8\pi\rho J_x\sqrt{2K}}}$		
	IIa	$\Gamma = \frac{Q\sqrt[4]{2}}{c_p}\cdot\frac{1}{\sqrt{3\pi\rho J_x\sqrt{K}}}\left[\int_{-\infty}^{+\infty}F'\theta\,d\varphi\right]^{-1}$	IIб	$(\Gamma = T_1 - T_2)$
7	I	$J_x = 2\pi x\int_{-\infty}^{+\infty}\rho u^2\,dy = \text{const},\ M_y = 2\pi x^2\int_{-\infty}^{+\infty}\rho uw\,dy = \text{const}$		
	IIa	$Q = 2\pi x\int_{-\infty}^{+\infty}\rho c_p u(T - T_\infty)\,dy = \text{const}$	IIб	—
8	I	$F''' + 2(FF')' = 0;\ \Phi'' + 2(F\Phi)' = 0$		
	IIa	$\theta'' + 2Pr_T(F\theta)' = 0$	IIб	$\theta'' + 2Pr_T F\theta' = 0$
9	I	$\varphi_{1/2} = 0{,}88$		
10	I	$F(0) = 0;\ F'(0) = \Phi(0) = 1;\ F''(\pm\infty) = \Phi(\pm\infty) = 0$		
	IIa	$\theta'(0) = 0;\ \theta(\pm\infty) = 0$	IIб	$\theta(+\infty) = 1,\ \theta(-\infty) = 0$
11	I	$F = \text{th}\,\varphi,\ F' = 1 - \text{th}^2\varphi;\ \Phi = F' = 1 - \text{th}^2\varphi$		
	IIa	$\theta(\varphi) = (F')^{Pr_T} = (\text{ch}\,\varphi)^{-2Pr_T}$	IIб	$\theta(\varphi) = \left[\int_{-\infty}^{\varphi}(\text{ch}\,\varphi)^{-2Pr_T}d\varphi\right]\left[\int_{-\infty}^{+\infty}(\text{ch}\,\varphi)^{-2Pr_T}d\varphi\right]^{-1}$
12	I	$G \sim x,\ E \sim x^{-1},\ \nu_T = KA = \text{const}\quad\left(K = \frac{a^2}{2}\right)$		
	IIa	$a_T = K_q A = \text{const}$	IIб	$a_T = K_q A = \text{const}\quad\left(Pr_T = \frac{K}{K_q}\right)$

A) Plane source jet; C) parallel or antiparallel flows; D) axisymmetric source jet; E) fan-type source jet.

If $\frac{x}{d} \gg 1$ at the same time $\frac{x}{x_0} \gg 1$, i.e., $x \approx \bar{x}$, since x_0 and d are quantities of the same order of magnitude.

In this connection we shall briefly touch the problem of the determination of the empirical constant a. Its value can be found from the experimental velocity distributions in the cross sections of the jet and along its axis for source jets. For the jet boundary, it is of course sufficient only to use the distribution in the jet cross section since here, as also in the case of $\frac{x}{x_0} \gg 1$, the coordinates x and x_0 are virtually coincident.

In the case of $x \approx x_0$ from the experimental velocity distributions in the jet cross sections we must select the corresponding values of $y_{1/2}$ and $x_{1/2}$, the coordinates of the points at which $\frac{u}{u_m} = \frac{1}{2}$. In each solution a certain characteristic "half" value of $\varphi_{1/2}$, corresponds to the value of $\frac{u}{u_m} = \frac{1}{2}$ which, for concrete flows, is given in Table 11.1. The constant a is then obtained from the formula

$$a = \frac{1}{\varphi_{1/2}} \frac{y_{1/2}}{x_{1/2}}.$$

In the case where the pole distance x_0 is equal to the distance $\bar{x}$ we can, using the transverse and longitudinal velocity distributions, determine the two empirical quantities x_0 and a which permit an approximate correspondence of the calculated distribution of the equivalent source jet and a real jet of finite dimensions.

For this purpose, in addition to the formula $a = \frac{1}{\varphi_{1/2}} \frac{y_{1/2}}{(\bar{x} + x_0)_{1/2}}$ in the domain of existence of an universal distribution, the empirical function $\frac{u_m}{u_0} = f(\bar{x})$, must be used which is represented in the form of $\frac{u}{u_m} = \frac{A}{(\bar{x} + x_0)^n}$. Since the constant A comprises the constant a (see Table 11.1) these two expressions permit the determination of both unknowns

x_0 and a.

The experiment shows that the approximate existence of universal distributions of velocity, surplus temperature, etc., are very general properties of turbulent jets. This justifies to a well-known degree, the application of the assumed universality of distributions at distances of $\bar{x}$ commensurable with x_0 in practical calculations.

To obtain an approximate idea, we give the mean values of the empirical constant a in the method of the asymptotic layer and the analogous constant a_k for the system of the finite layer [4].

For the jet boundary $a \approx a_k \approx 0{,}09 \div 0{,}12$; for the plane source jet $a \approx 0{,}10 \div 0{,}12$, correspondingly $a_k \approx 0{,}09 - 0{,}11$; for a round jet $a \approx 0{,}045 \div 0{,}055$, correspondingly $a_k \approx 0{,}066 - 0{,}08$. Higher values of the constants correspond to nonuniform initial velocity distributions; the consideration of thi nonuniformity has been discussed by G.N. Abramovich [4, 11].

The so-called pole distance x_0 is still less stable. We mentioned already that it may be both positive (pole in front of the nozzle section) and negative (pole behind the nozzle section). In G.N. Abramovich's last papers [11] the interesting attempt has been made to replace in the case of "usual" conditions the two constants (x_0/d and a) by a single constant $C \approx 0{,}22$, which enters the formula for the width of the jet in the form $b = Cx$ or $\frac{db}{dx} = C$. This constant applies satisfactorily to various jets (plane and round, submerged and parallel) but, of course, it is not really universal. This holds true in particular for the values of the empirical constants of the type of a (or a_k), and also for the value of the pole distance x_0/d. The latter is due to the fact that the flow has been divided into two sections (the initial, corresponding to the jet boundary, and the fundamental, corresponding to the source jet) or into three sections (including an intermediate one). These methods are not considered here since in the following

we shall discuss a more general method which permits the calculation of the continuous, nonself-similar deformation of the initial distributions (of velocity, temperature, ρu^2) up to the transition to the self-similar flow for a source jet (see Chapter 14).

Thus, the empirical constant a entering the calculation according to the method of the asymptotic layer is by no means a physical constant. It is strongly influenced by factors which are difficult to take into account such as the characteristics of the pulsatory microstructure of the jet (mainly, but not only, the initial intensity of pulsations, the initial level of turbulence), the nonuniformity of the initial distributions, and also the temperature and the like.

In cases different from the "usual" one the value of a and, together with it, also the intensity of damping of the jet, vary considerably. For example, even in 1935, in a paper by D.N. Lyakhovskiy and S.N. Syrkin [136] a value of $a_k \approx 0{,}27$ was given for a twisted axisymmetric jet and this differs assentially from the value $a_k \approx 0{,}07$ for "usual" conditions. In our own experiments a variation of the initial pulsation level from about 1.5 to 4.5% was accompanied by a variation of the value of a_k from about 0.065 to 0.08. In order to illustrate in this case the great differences between the real dependences of u_m/u_0 on $\bar{x}/d$ we point out that under these conditions and for a strictly constant initial velocity distribution the value of $\bar{x}/d = 10$ amounts to 0.43 and 0.28, respectively (for comparison we give the value calculated according to G.N. Abramovich [11]: with $C = 0.22$ it is equal to $u_m/u_0 = 0.38$).

This problem (on the variation of the value of the constant a) is also considered when investigating gas jets. Altogether, it can be shown that with an appropriate choice of the empirical constant, the self-similar solution is a good approximation to real turbulent jets if the inequality $\frac{x}{d} \approx \frac{x}{x_0} \gg 1$ is satisfied.

Deviations from the latter, which are valuable in certain applications, are always linked with a reduction of accuracy and a series of additional requirements, the necessity of taking the initial conditions into account, of the nonuniformity of the initial distributions, the level of turbulence, and the like.

11.3. THE MIXING OF PARALLEL AND ANTIPARALLEL FLOWS

In order to solve the problem of the turbulent mixing zone which appears at the boundary of two parallel or antiparallel flows (co-moving or counterflows) we follow papers [51, 60], and use Prandtl's second formula for the kinematic coefficient of the turbulent viscosity

$$\nu_T = K(u_1 - u_2)x, \tag{11.1}$$

and, taking this into account, we can rewrite the system of equations of the plane turbulent boundary layer:

$$\left.\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= K(u_1 - u_2)x\frac{\partial^2 u}{\partial y^2}, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0. \end{aligned}\right\} \tag{11.2}$$

To this system, we add the equation of heat propagation

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = K_q(u_1 - u_2)x\frac{\partial^2 \theta}{\partial y^2} \tag{11.3}$$

(in these equations u_1 and u_2 denote the velocities of the uniform flows, $\theta = \frac{T - T_2}{T_1 - T_2}$, where T_1 and T_2 are the temperatures of these flows, K and K_q are constants to be determined by way of experiment).

The boundary conditions for the functions $u(x, y)$and $\theta(x, y)$ have the same form as with the analogous problem of the mixing of laminar flows:

$$\left.\begin{aligned} &u = u_1, \ \frac{\partial u}{\partial y} = 0, \quad \theta = 1 \quad \text{if} \quad y = +\infty, \\ &u = u_2, \qquad\qquad \theta = 0 \quad \text{if} \quad y = -\infty. \end{aligned}\right\} \tag{11.4}$$

Let us put

$$\frac{u}{u_1} = F'(\varphi), \quad \theta = \theta(\varphi), \quad \varphi = \frac{y}{ax}; \tag{11.5}$$

we then obtain instead of the system of equations (11.2) and (11.3) in partion derivatives the following system of ordinary differential

equations:

$$F''' + 2FF'' = 0, \qquad (11.6)$$

$$\theta'' + 2\,\mathrm{Pr}_T F\theta' = 0 \qquad (11.7)$$

(here we used the additional relations:

$$a^2 = 2K(1-m), \quad \mathrm{Pr}_T = \frac{K}{K_q}, \quad m = \frac{u_2}{u_1}\Big).$$

These equations, together with the boundary conditions

$$\left.\begin{array}{llll} F' = 1, & F'' = 0, & \theta = 1 & \text{if} \quad \varphi = +\infty, \\ F' = m, & & \theta = 0 & \text{if} \quad \varphi = -\infty, \end{array}\right\} \qquad (11.8)$$

agree with the corresponding equations and boundary conditions for the mixing of laminar flows given in Section 5.3. The expressions for $F'(\varphi)$ and $\theta(\varphi)$ will therefore have the same form:

$$F' = \frac{u}{u_1} = 1 + \frac{1}{2}(m-1)(1 - \operatorname{erf}\varphi), \qquad (11.9)$$

$$\theta = \frac{1}{2}[1 + \operatorname{erf}(\varphi\sqrt{\mathrm{Pr}_T})]. \qquad (11.10)$$

The expressions for the transverse velocity components and the stream-functions will differ from the corresponding expressions in the laminar problem owing to the different form of the variable φ:

$$\frac{v}{au_1} = \varphi F' - F = \frac{1}{2}(m-1)\left\{\int_0^\varphi (\operatorname{erf} z)\,dz - \varphi \operatorname{erf}\varphi - C_2\right\}, \qquad (11.11)$$

$$\frac{\psi}{u_1 a x} = \varphi + \frac{1}{2}(m-1)\left\{\varphi - \int_0^\varphi (\operatorname{erf} z)\,dz + C_2\right\}. \qquad (11.12)$$

Equations (11.9), (11.11) and (11.12) can be rewritten in the following form:

$$\frac{u-u_2}{u_1-u_2} = \left(\frac{u}{u_1}\right)_{m=0} = \frac{1}{2}(1 + \operatorname{erf}\varphi), \qquad (11.13)$$

$$\frac{v}{a(u_1-u_2)} = \frac{v}{(au_1)_{m=0}} = \frac{1}{2}\left\{\varphi \operatorname{erf}\varphi - \int_0^\varphi (\operatorname{erf} z)\,dz + C_2\right\}, \qquad (11.14)$$

$$\psi = \psi_2 + \frac{u_1-u_2}{u_1}\psi_{m=0} = u_2 y + \frac{1}{2}a_{m=0}u_1 x(1-m) \times$$

$$\times\left\{\varphi - \int_0^\varphi (\operatorname{erf} z)\,dz - C_2\right\}. \qquad (11.15)$$

The difference between the righthand sides of Eqs. (11.13)-(11.15) and the corresponding equations of the laminar motion are caused by the

difference in the structure of φ and the factor a entering φ. The first of the latter facts (i.e., the transition from $\varphi = \frac{By}{\sqrt{x}}$ to $\varphi = \frac{y}{ax}$) affects the dependence of v on ψ and on x. As to the second ($a = \sqrt{2K(1-m)}$ instead of $B = \frac{1}{2}\sqrt{\frac{u_1}{\nu}}$), it corresponds to the initial physical model of turbulent mixing.

It is essential that, by virtue of the assumption of the proportionality of the coefficient of turbulent viscosity to the difference $u_1 - u_2$, the quantity a depends on m:

$$\frac{a}{a_{m=0}} = \sqrt{1-m}. \qquad (11.16)$$

The latter indicates in particular that the effective zone of the turbulent boundary layer in the plane xOy tends to zero in the absence of anisotropic turbulent mixing of parallel wakes of equal velocities.

Let us now turn to the problem of the agreement of the solution obtained and the experiment.

Since for the particular case of $m = 0$ we have sufficiently reliable experimental data at our disposal [4, 11] etc., it is natural that we use them in order to "bind" the solution obtained. With this we mean firstly the agreement between calculated and experimental velocity distributions and secondly the agreement between calculated and experimental positions of the neutral streamlines.

Considering the first problem, the distribution of the longitudinal velocity component u, we see that from Eq. (11.9) with $m = 0$ and $\varphi = 0$ it follows that $\frac{u}{u_1} = 0{,}5$; with these conditions the experiment [4] yields $\frac{u}{u_1} \approx 0{,}68$. In order to obtain mathematical results which agree with the experiment, we use the above property of the boundary layer equations. On the basis of this property we can write the solution of Eq. (11.6) with $m = 0$ which satisfies the boundary conditions and the equation $\frac{u}{u_1} \approx 0{,}68$ with $\varphi = 0$ in the form

$$\frac{u}{u_1} = F'(\varphi^*) = F'(\varphi + 0{,}33) = \frac{1}{2}[1 + \mathrm{erf}(\varphi + 0{,}33)]. \qquad (11.17)$$

We generalize this solution for the case of $m \neq 0$:

$$\frac{u-u_2}{u_1-u_2} = F'(\varphi^*) = \frac{1}{2}\,[1 + \operatorname{erf}(\varphi + 0{,}33)]. \qquad (11.18)$$

Now we pass over to the problem of the stream-function. Rewriting Eq. (11.12) taking into account Eq. (11.18) in the form

$$\frac{\psi}{a u_1 x} = \varphi + \frac{1}{2}(m-1)\left\{\varphi + 0{,}33 - \int_0^{\varphi+0{,}33} (\operatorname{erf} z)\,dz + C_2\right\}, \qquad (11.19)$$

we see that agreement with the experiment ($\psi = 0$ with $\varphi = -0.185$ if $m = 0$ [51]) is obtained with $C_2 = -0.5$*).

The expression for the streamfunction will therefore take the form

$$\frac{\psi}{a u_1 x} = \varphi + \frac{1}{2}(m-1)\left\{\varphi - \int_0^{\varphi+0{,}33} (\operatorname{erf} z)\,dz - 0{,}17\right\} \qquad (11.20)$$

or with $m = 0$

$$\left(\frac{\psi}{a u_1 x}\right)_{m=0} = \frac{1}{2}\left\{\varphi + \int_0^{\varphi+0{,}33} (\operatorname{erf} z)\,dz + 0{,}17\right\}. \qquad (11.21)$$

The transverse velocity component will be equal to

$$\left(\frac{v}{a u_1}\right)_{m\neq 0} = \frac{1-m}{2}\left\{\varphi \operatorname{erf}(\varphi + 0{,}33) - \int_0^{\varphi+0{,}33} (\operatorname{erf} z)\,dz - 0{,}17\right\} \qquad (11.22)$$

or, with $m = 0$, to

$$\left(\frac{v}{a u_1}\right)_{m=0} = \frac{1}{2}\left\{\varphi \operatorname{erf}(\varphi + 0{,}33) - \int_0^{\varphi+0{,}33} (\operatorname{erf} z)\,dz - 0{,}17\right\}. \qquad (11.23)$$

Figure 11.2 shows the distributions of the velocity u according to Eq. (11.18) and of the velocity v according to Eq. (11.23).

As we have no reliable experimental data for the case of $m \neq 0$ at our disposal we assume for the following construction that $C_2 = -0.5$ within the range of values of m. This assumption must of course be verified experimentally and if necessary another value must be taken. In any case, it does not cause distortions in the qualitative flow patterns considered below.

In order to judge the flow pattern we establish first of all the characteristic values $\varphi_{v=0}$ and $\varphi_{\psi=0}$, corresponding to the vanishing of

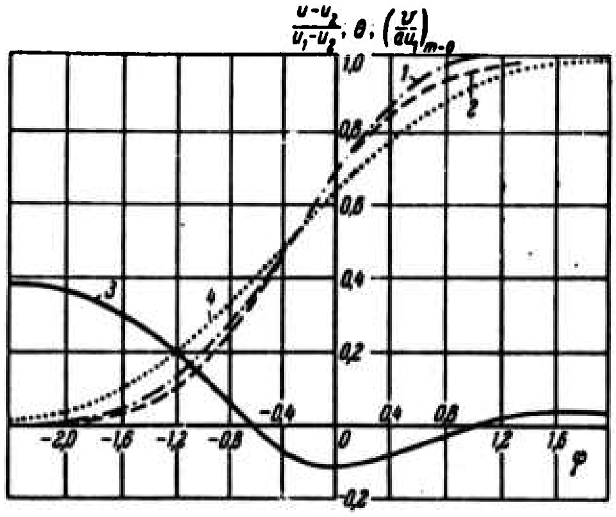

Fig. 11.2. Reduced distributions of longitudinal and transverse velocities and temperature (calculated). 1) Longitudinal velocity component according to data [4, 317]; 2) longitudinal velocity component; 3) transverse velocity component; 4) temperature.

the flow velocity u and the stream function ψ respectively, with different values of the parameter m.

The results of the solution (obtained by graphical analysis from the equations $F' = 0$ and $\psi = 0$) are represented in Fig. 11.3 which shows the characteristic angles as functions of the parameter m. The flow pattern corresponding to this graph can be represented in all regions of variation of m in the form of a successive shift of the flow patterns shown in Fig. 11.4 a-e (see also Fig. 11.5).

The first three patterns of Fig. 11.4 ($0 \leqslant m \leqslant 1$) refer to parallel motions whose nature is shown by the figures.

The fourth case (Fig. 11.4d) corresponds to a relatively weak counterflow which is accompanied by a partial inversion of the motion enclosed in a certain domain between the two neutral streamlines $\psi = 0$ on either side of the ray on which the longitudinal velocity component vanishes.

The last pattern (Fig. 11.4f) corresponds to a very strong coun-

terflow where the flow as a whole behaves as if it were streaming around a dividing wall. The transient form between these two flow patterns is the case of "sliding" streams (see Fig. 11.4e).

Note that these flow patterns were obtained previously in paper [60] by way of constructing a solution "excessive" with respect to the momentum, which generalizes the result by Tollmien-Abramovich for the case of $m = 0$.

It is characteristic that in certain cases (Fig. 11.4e,f,) the calculation indicates the establishment of a circulating inverse motion of the fluid in the field of constant pressure, which is caused by the turbulent viscosity (and in the case of laminar motion, for which the flow pattern is the same, it is caused by the molecular viscosity). A more accurate calculation (of the inverse flow) would require the consideration of the pressure distribution caused by the curvature of the streamlines.

As the above investigation was based on the supposition that the pressure was constant throughout the field of flow, it is not uninteresting to try an approximate estimation of the pressure differences in the two streams (at points which are far away from one another).

For this purpose, we make use of W. Tollmien's method [317] which consists of applying the second equation of motion (for the transverse velocity component) which had not been taken into account previously:

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \frac{1}{\rho}\frac{\partial \tau_T}{\partial x}.$$

Assuming that in a zero approximation $\frac{\partial p}{\partial y} = 0$, we obtain for the first approximation

$$\frac{\partial p}{\partial y} = -2\frac{\partial \tau_T}{\partial x}.$$

Using the expression $\tau_T = \rho K(u_1 - u_2)x\frac{\partial u}{\partial y}$ we obtain

$$\frac{\partial \tau_T}{\partial x} = \rho K u_1(u_1 - u_2)\frac{\partial}{\partial y}(F' - \varphi F''),$$

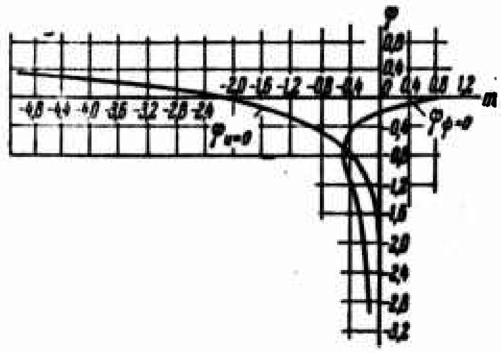

Fig. 11.3. The characteristic values of the coordinates $\varphi_{\psi=0}$ and $\varphi_{u=0}$ as functions of the parameter m.

and consequently

$$\left[\frac{p}{\rho}\right]_{y_1}^{y_2} = -2u_1(u_1 - u_2)K[F' - \varphi F'']_{\varphi_1}^{\varphi_2}.$$

Thus, with $y_1 \to -\infty$, $y_2 \to +\infty$ we have

$$\frac{p_{\infty} - p_{-\infty}}{\rho} = -2K(u_1 - u_2)^2$$

or finally

$$\frac{p_{\infty} - p_{-\infty}}{\left[\frac{\rho(u_1 - u_2)^2}{2}\right]} = -4K.$$

A numerical estimation [250] for $m = 0$ yields $4K \approx 0{,}0055$. If we assume that the coefficient K depends only slightly on m, the absolute pressure difference will increase as $|m|$ increases in the range of $m < 0$. But within the limits of variation of m which are of practical interest, the influence of pressure variations is negligibly small in the case of parallel flows. As regards the counterflows, we must expect for them an essentially higher influence of the pressure distribution, though they are hardly investigated in experiments.

The solution of the thermal problem from the differential equation (11.7) and the boundary conditions (11.8) for a turbulent motion agrees with the analogous solution for the laminar motion. We may there-

fore write it in the form

$$\theta(\varphi) = \frac{1}{2}\{1 + \operatorname{erf}[(\varphi + 0{,}33)\sqrt{Pr_T}]\}. \qquad (11.24)$$

Figure 11.2 shows the temperature distribution curve in the mixing zone, which was also calculated from Eq. (11.24).

As we see from the figure, the conventional boundaries of the thermal boundary layer are somewhat wider than the boundaries of the dynamic layer, a fact which is in agreement with the experiment. Just as in the case of the velocity distribution, the influence of the parameter m is taken into account by the constant a.

In this way, the problem of the turbulent mixing of two uniform flows is solved finally within the framework of the asymptotic layer method for both parallel and antiparallel motions.

Let us briefly consider the solution of this problem by means of the method of the layer of finite thickness.

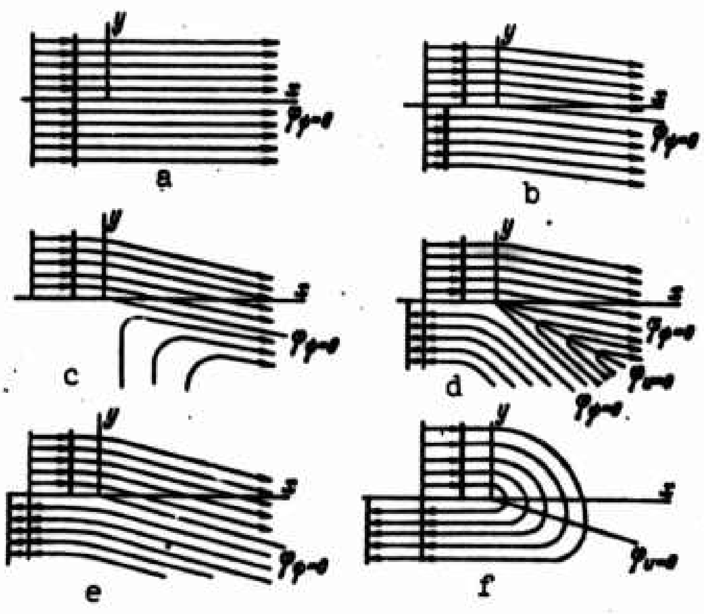

Fig. 11.4. Flow patterns of parallel and antiparallel motions: а) $m = 1$; б) $0 < m < 1$; в) $m = 0$; г) $-0{,}5 < m < 0$; д) $m = -0{,}5$; е) $-\infty < m < -0{,}5$.

A. Kuethe's paper [258] contains a generalization of W. Tollmien's problem on the jet boundary ($m = 0$) to the range of parallel flows ($0 \leqslant m \leqslant 1$). Let us consider the possibility of extending this solution to the range of negative values of m [51].

Using Prandtl's first formula $\nu_T = c^2 x^2 \frac{\partial u}{\partial y}$ and assuming that $u = u_1 F'(\varphi)$, $\varphi = \frac{y}{ax}$, and also $\frac{T - T_2}{T_1 - T_2} = \theta(\varphi)$ and $a_T = c_T^2 x^2 \frac{\partial u}{\partial y}$, we obtain instead of the system of equations (11.6) and (11.7) the following system (with $a = 2c^2$ and $\Pr_T = \frac{c^2}{c_T^2}$):

$$F''(F''' + F) = 0 \tag{11.25}$$

and

$$(F''\theta')' + 2\Pr_T F\theta' = 0. \tag{11.26}$$

First of all, we shall consider Eq. (11.25) which can be split into two differential equations:

$$F''(\varphi) = 0, \tag{11.27}$$

$$F'''(\varphi) + F(\varphi) = 0. \tag{11.28}$$

Equation (11.27) is of no interest as it describes the flow in the zone of constant velocity ($u = u_1 F' = \text{const}$).

The boundary conditions for Eq. (11.28) read

$$\left.\begin{array}{lll} F' = 1, & F'' = 0 & \text{if} \quad \varphi = \varphi_1, \\ F' = m, & F'' = 0 & \text{if} \quad \varphi = \varphi_2. \end{array}\right\} \tag{11.29}$$

The fifth boundary condition, which is necessary tn order to determine the three constants of integration and the two boundaries of the mixing zone φ_1 and φ_2 is in paper [235] given in the two variants

$$F(\varphi_1) = \varphi_1 \quad (v = 0) \text{ with } \varphi = \varphi_1 \tag{11.30}$$

or

$$m = \frac{F(\varphi_1) - \varphi_1}{m\varphi_2 - F(\varphi_2)} \text{ with } \varphi = \varphi_1. \tag{11.30a}$$

In the case of $m = 0$ condition (11.30a) is replaced by (11.30).

The first of these conditions has been used by Tollmien [317] for $m = 0$; the second one is T. Karman's condition ($u_1 v_1 + u_2 v_2 = 0$) the condition for the absence of external forces acting perpendicularly

to the main direction of motion. Note by the way that these conditions are both obtained from the integral relation of momenta

$$\int_{y_2}^{y_1} \rho u^2 dy = \rho\,(y_1 u_1^2 + y_2 u_2^2).$$

When we use (11.29) and the inequality $F(\varphi_2)F'(\varphi_2) < 0$ to represent the integral of the lefthand side of the latter equation in the form

$$\int_{\varphi_2}^{\varphi_1} F'^2 d\varphi = F(\varphi_1)\,F'(\varphi_1) + |\,F(\varphi_2)\,F'(\varphi_2)\,|,$$

we obtain finally (11.30a):

$$F(\varphi_1) = \varphi_1 + m^2\varphi_2 - mF(\varphi_2).$$

Thus we see that both conditions (11.30) and (11.30a) are contained in the integral condition of momentum conservation for the cases $m = 0$ and $m \neq 0$, respectively. It is obvious that (11.30) cannot be applied if $m \neq 0$.

Introducing $\bar{\varphi} = \varphi - \varphi_1$, we can represent the solution to Eq. (11.28) in the form

$$F(\bar{\varphi}) = C_1 \exp(-\bar{\varphi}) + \exp\left(\frac{\bar{\varphi}}{2}\right)\left\{C_2 \cos\frac{\sqrt{3}}{2}\bar{\varphi} + C_3 \sin\frac{\sqrt{3}}{2}\bar{\varphi}\right\}. \quad (11.31)$$

Using the boundary conditions (11.29) we can find the constants of integration C_1, C_2, C_3 and the value of $\Delta\varphi = \varphi_2 - \varphi_1$ as functions of m. The quantity $\Delta\varphi$ is then a transcendential function of m:

$$m = \frac{\exp\left(\frac{1}{2}\Delta\varphi\right) + \exp(-\Delta\varphi)\left(\sqrt{3}\sin\frac{\sqrt{3}}{2}\Delta\varphi - \cos\frac{\sqrt{3}}{2}\Delta\varphi\right)}{\cos\frac{\sqrt{3}}{2}\Delta\varphi + \sqrt{3}\sin\frac{\sqrt{3}}{2}\Delta\varphi - \exp\left(-\frac{3}{2}\Delta\varphi\right)}. \quad (11.32)$$

Figure 11.5 shows a graph of $\Delta\varphi$ as a function of m from which we see that a solution of the problem, which satisfies Condition (11.29), only exists in the range $-0.21 < m < 1$. Moreover, this solution is not unambiguous*).

It is natural that in the range of values of m between zero and one the solution obtained by the method of the asymptotic layer must

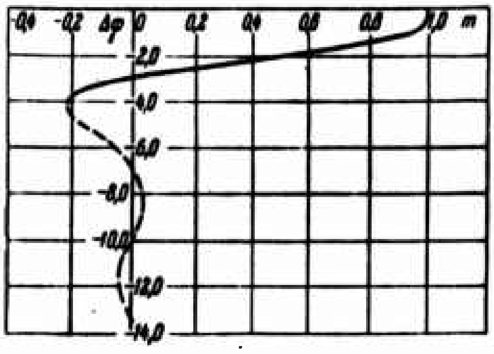

Fig. 11.5. Dependence of $\Delta\varphi = \varphi_2 - \varphi_1$ on the parameter *m* (calculated according to the finite-layer method.

be relatively close to the results of paper [258]; as to the range of antiparallel motions $(-0.21 \leqslant m \leqslant 0)$ it corresponds to the flow pattern shown in Fig. 11.4,d (if we again restrict ourselves to the maximum value of $\Delta\varphi$). The solution of the thermal problem obtained by integrating Eq. (11.26) with the boundary conditions

$$\left.\begin{array}{l} \theta = 1 \text{ with } \varphi = \varphi_1, \\ \theta = 0 \text{ with } \varphi = \varphi_2, \end{array}\right\} \quad (11.33)$$

is represented by a straight line (with discontinuities of the heat flux values at the boundaries of the mixing zone):

$$\theta = \frac{\varphi - \varphi_2}{\varphi_1 - \varphi_2}. \quad (11.34)$$

We think that the impossibility of generalizing the solution [258] to the entire domain of negative values of *m* and also the ambiguousness of the solution and, finally, the linearity of the temperature distribution (11.34) proves that the method of the layer of finite thickness is artificial also with this problem. As shown above, the method of the asymptotic layer does not display these failings.

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1, 3, 4, 6, 10, 11, 13, 25, 51, 52, 56, 58, 60, 75, 77, 79, 85, 87, 114, 130, 133, 134, 161, 174, 193, 194, 212, 213, 214, 216, 250, 258, 281, 282, 311, 317, 319, 321, 323, 325.

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268	Note that the determination of the coordinate φ, corresponding to $\psi = 0$ with $m = 0$, immediately from the velocity distribution (11.9) through an integration of the flow rate, would yield $\varphi_{\psi=0} = -0{,}23$ $C_2 = -0.55$; if, however, the following approximations are taken into account in the velocity distribution, the integration results in the same experimental value $\varphi_{\psi=0} = -0{,}185$, as accepted in the text.
274	The innumerable multitude of solutions, however, corresponds to the problem of the edge of the jet, since for the latter $\Delta\varphi$ is a solution to the transcendental equation $\exp(3/2\,\Delta\varphi) + \sqrt{3}\sin(\sqrt{3}/2)\Delta\varphi - \cos(\sqrt{3}/2)\Delta\varphi = 0$. W. Tollmien [317] and later on also other authors (A.Kuethe) chose the first root of this equation (which is in agreement with experiment) without mentioning the existence of other roots.

Chapter 12

THE SEMILIMITED TURBULENT JET

12.1. THE SCHEME OF A SEMILIMITED JET

A semilimited jet is the result of the efflux of a viscous fluid and its expansion along a solid surface in an unlimited space filled by a fluid at rest. The experiment shows that the flow in such a jet near the nozzle becomes turbulent for the most part. This is true in the first place for the outer part of the jet. In the boundary layer near the wall, as usually, in a motion along the wall a laminar layer may develop at first, which, at a certain distance from the orifice of the nozzle, becomes turbulent. The laminar flow is then maintained in a very thin laminar sublayer along the wall (see diagram of Fig. 12.1).

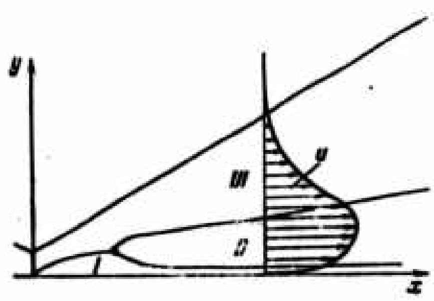

Fig. 12.1. Schematic representation of the boundary layer of a turbulent semilimited jet. I – Laminar layer ($\nu_I = \nu$); II – turbulent layer along the wall ($\nu_{II} \sim y u_m$); III – free turbulent layer ($\nu_{III} \sim x u_m$).

The motion of the fluid in the boundary layer formed by the semilimited jet is, as we see from this description, very complex: various forms of boundary layers coexist in it, those adjacent to the wall (laminar or turbulent) with the free jet layers. When the Reynolds

numbers Re_x are not too high (as calculated from the maximum velocity and distance from the nozzle orifice) the flow may, however, even be purely laminar since the presence of the solid surface exerts a stabilizing influence not only on the flow in the zone adjacent to the wall but also on the motion in the external part of the jet, though there it is much weaker. Thus, in Z.B. Sakipov's experiments [66], with values of Re = 650 for the Reynolds number of the issuing jet, the motion in an oil jet, streaming along a round rod, remained laminar. Without this rod, the flow near the orifice became mainly turbulent in the free jet, even with Re numbers of the order of several tens. (With comparatively small values of Re a considerable part of the flow near the nozzle remains approximately laminar even in a free jet.) With respect to the order of magnitude of the Reynolds number calculated from the initial velocity and the characteristic dimension of the nozzle, these observations agree with the data of paper [224] for the free jet.

As regards the turbulization of the boundary layer adjacent to the wall, for it (besides the initial conditions), as in the case of a uniform flow around a plate, we have the Reynolds number Re_x as the criterion, which is calculated from the nozzle distance and the characteristic (maximum) velocity in the jet cross section. The latter, as we see from the solution of the laminar problem, decreases as the nozzle distance increases, the number Re_x, however, increases (since the product $u_m x \sim x^{\alpha+1}$ increases with increasing x and $|\alpha| < 1$). This remark is only true for a plane-parallel semilimited jets, as for a fan-type jet it may happen that $|\alpha| > 1$.

The complex structure of the boundary layer is the reason for the difficulties of predicting the fundamental laws of jet expansion. Nonetheless, we shall try to consider the problem on the basis of a dimension analysis which applied to a plane-parallel semilimited jet.

The outer zones of the jet, as in all other cases of an evolved turbulent motion, must obey the general laws for self-similar jet flows, i.e., the outer boundaries must be rectilinear ($\beta = -1$). This indicates that the ordinate $y_{1/2}$ for the outer part of the jet increases proportionally to the coordinate x. Recall that $y_{1/2} \sim x^{-\beta} \sim x^{3/4}$ for a laminar plane, semilimited jet. In the section where the wall layer is laminar, its thickness (characterized by the value of the coordinate $y_{1/2}$ for the inner part of the jet and, approximately, by the value of the ordinate, y_m, which corresponds to the maximum velocity) must grow more slowly: for a laminar semilimited jet $\beta = -\frac{3}{4}$, while $\beta = -\frac{1}{2}$ in the case of a uniform laminar flow streams around a plate.

We base our reasoning about a probable law of variation of the maximum velocity u_m on the rough supposition that the coefficient v_T of turbulent kinematic viscosity is constant in a cross section of the jet. In this case [with $\nu_T = \nu_T(x)$] the integral invariant conserved for the motion has the form

$$K = \int_0^\infty \rho u^2 \left(\int_0^y \rho u \, dy \right) dy = \text{const}$$

or the simpler invariant for the self-similar flow (the particular case of Eq. (10.8) with $\sigma = \frac{3}{2}$ for the semilimited jet) of the form

$$P = \int_0^\infty u^{3/2} dy = \text{const}.$$

With $\beta = -1$ we hence obtain $\alpha = -\frac{2}{3}$ and $\nu_T \sim x^{1/3}$.

As shown previously, for a laminar jet $\alpha = -\frac{1}{2}$. It is obvious that for the section of a laminar wall layer we must expect a variation of the maximum velocity according to laws similar to those of the laminar jet, and after turbulization — to those of the turbulent jet. In this analysis, we did not take into account the part played by the laminar sublayer adjacent to the wall and the turbulent wall layer in which

$v_T \sim yu_m$. This simplification does not exert an essential influence on the integral characteristics of the jet, as this can be concluded from experimental data.

For a semilimited fan-type jet (and also the turbulent flow along a cone) we give the value of the constant α of self-similarity only for an evolved turbulent motion in which the influence of the turbulent wall layer has been neglected. To this motion corresponds a value of the constant of self-similarity of $\alpha = -\frac{4}{3}, \beta = -1$, and for a slightly twisted jet $\varepsilon = -\frac{7}{3}$ (for the peripheral velocity) and $\delta = -\frac{14}{3}$ (for the pressure distribution in a jet flowing along a cone).

It will be shown that experimental investigations of a semilimited turbulent jet verify the above considerations on the two sections of the flow, i.e., before and after the turbulization of the wall layer in a plane jet.

12.2. TURBULIZATION OF THE BOUNDARY LAYER ADJACENT TO THE WALL

Following paper [66], we shall consider briefly some experimental data on the transition of the laminar form of motion to the turbulent form in the boundary layer adjacent to the wall, of a semilimited jet. Note that in experiments by F. Förtmann [244], and in later experiments by P. Bakke [226] and A. Sigalla [297] such a transition could not be observed*).

Its presence was proven by a systematic analysis of the curves in graphs showing $y_{\frac{1}{2}}$ and u_m as functions of the coordinate x in A.T. Trofimenko's experiments [191, 192]. In order to explain the apparent contradiction special measurements were made with an air jet and also (Z.B. Sakipov) experiements in various devices with semilimited jets of air and water. The results of these experiements, which are described in detail in paper [66], showed that turbulization of the boundary layer adjacent to the wall may occur, according to the conditions of efflux,

either immediately at the orifice of the nozzle or at a considerable distance away from it, or this effect may even be absent in entire region of the jet that was under observation.

The first case where the flow in the whole space occupied by the jet is turbulent, except for the laminar sublayer, was observed as a rule with fan-type semilimited jets and plane jets, when the outlet nozzle had been made without a sufficiently strong "draw-in." It is obvious that in these cases, the initial disturbances together with the expansion of the jet at the outlet result in a turbulization which sets in virtually immediately in the whole boundary layer.

The second type of flow with turbulization of the wall layer at distances of about 15-20 nozzle diameters away from the orifice at values of $Re_x \approx (3 \div 5)10^5$ occurs when a fluid issues out of a nozzle with strong "draw-in." The initial disturbances are here reduced to a minimum and the laminar sublayer which forms at the wall is maintained over a considerable distance.

Finally, the third type of flow, where the laminar wall layer extends over the whole range of the flow, can be observed when the efflux conditions are such that no initial disturbances arise and Re_x does not reach the critical value. This was the case in F. Förtmann's experiments [244] and, apparently, also in a paper by W. Schwarz and W. Cosart [296].

Figure 12.2 shows $y_{\frac{1}{2}}$ (for the outer part of the jet) as a function of the coordinate $\bar{x}$: for the first type of flow (water jet discharged from nozzle with slight "draw-in," initial speed $u_0 = 3{,}0$ m/sec) it is Fig. 12.2a, and for a flow with transition (air jet with $u_0 = 52$ m/sec and water jet with $u_0 = 3{,}2$ m/sec) it is Fig. 12.2b.

Figure 12.3 shows the integral characteristics (total momentum $\frac{J_x}{J_0}$ and the quantity $\frac{K}{K_0}$) along the jet for all three types of flow.

As we see from the figures, the presence or absence of a transition is expressed clearly by the form of the curves.

As regards the shape of the curves $J_x(x)$ and $K(x)$, it may be explained in the following way. The condition K = const refers to an evolved self-similar flow in a case where the coefficient ν_T does not vary across a section. This condition, however, is only satisfied in a purely laminar jet, whereas in the cases represented in Fig. 12.3, $\nu_T = \nu$ for the laminar wall layer $\nu_T \sim yu_m$ in the turbulent wall layer and $\nu_T \sim xu_m$ in the inner part of the jet (see schematic diagram of Fig. 12.1).

This can also explain the shape of the curves $K(x)$ in various sections of the jet: the noticeable increase of the quantity K if the wall

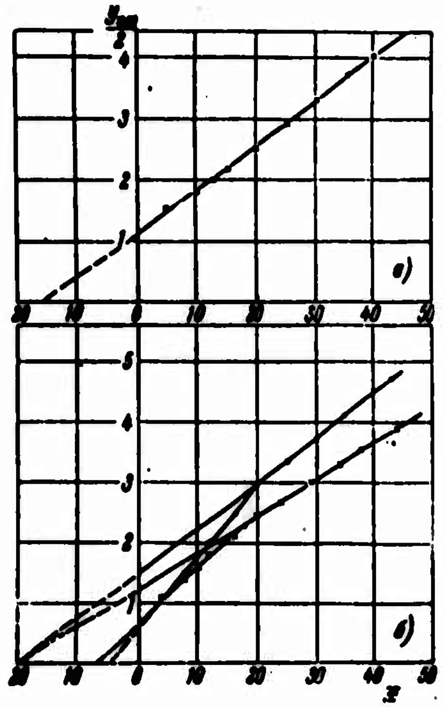

Fig. 12.2. Dependence of the conditional ordinate of a turbulent semilimited jet on the distance to the nozzle cross section. a) Water jet discharged from a nozzle with slight "draw-in;" b) water jet (++) and air jet (oo) discharged from a nozzle with strong "draw-in."

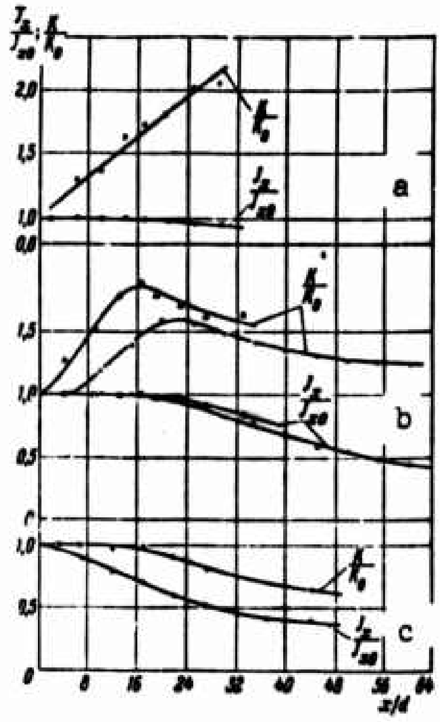

Fig. 12.3. Dependence of the relative values of the integral characteristics J_x/J_0 and K/K_0 from the distance to the nozzle cross section in a plane, turbulent semilimited jet. a) Air jet (experiments by F. Förtmann [244]; b) air jet (ΔΔ-□□) and water jet (××-oo) discharged from a nozzle with strong "draw-in" [66]; c) water jet discharged from a nozzle with weak "draw-in" [66].

layer is laminar and the slight drop when it is turbulent.

The value of the momentum remains almost constant in the case of a laminar flow in the wall layer, when friction is very small, and it decreases considerably when the boundary layer becomes turbulent.

Figure 12.4 shows the intensity distribution of velocity pulsations in an air jet moving along a plate measured by means of a thermoanemometer (1 mm away from the wall), analogous data for a water jet and, for comparison, the velocity pulsations of air in the turbulization zone of the boundary layer on a wing profile*).

These and other observations described in the papers referred to prove without doubt that the effect we discussed above must exist in

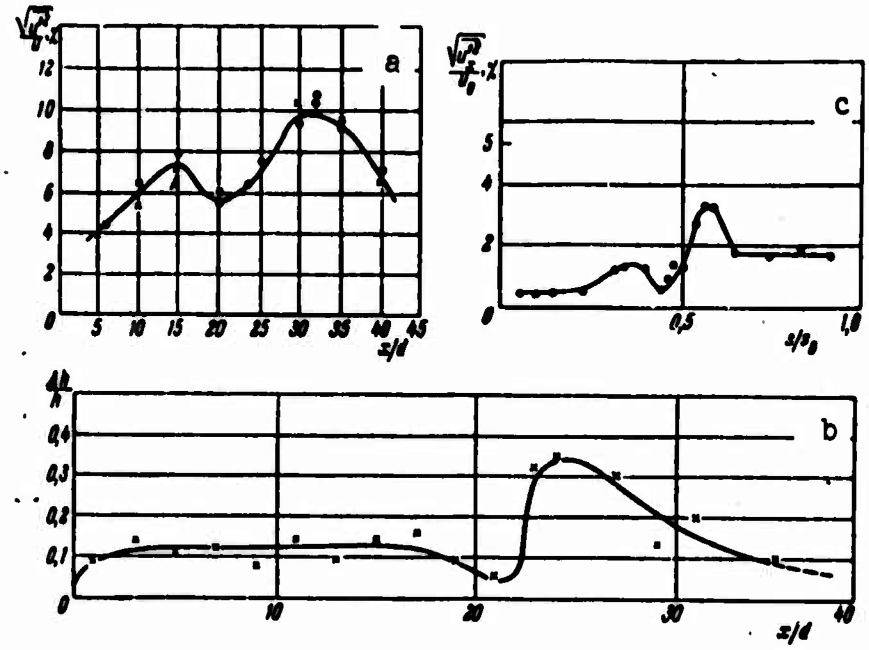

Fig. 12.4. Intensity distribution of velocity pulsations in a turbulent semilimited jet of air (a) and water (b) at a distance of 0.1 cm from the plate [66]; c) the same for a wing profile [130].
Note: For water (Fig. 12.4b) on the ordinate axis the pulsations of the fluid column of a micronometer are plotted.

a turbulent semilimited jet. Quantitative results, in particular the values of the constants of self-similarity also speak in favor of it. In all cases of a turbulent semilimited jet the constant of self-similarity was found to be equal to $\beta = -1$.

In experiments referring to the first type of flow (turbulent flow) the constant $-\alpha \approx 0{,}60 \div 0{,}62$; almost the same value ($-\alpha \approx 0{,}73 \div 0{,}78$ for an air jet and $-\alpha \approx 0{,}80 \div 0{,}83$ for a water jet) was observed in the second section, that is, after the transition, in the second type of flow. Recall that the calculation yields $-\alpha \simeq 0.67$ for $v_T = bu_m$.

Finally, in the third case of a laminar wall layer (throughout the jet or only in the first section of it) the value of the constant α amounted to $-\alpha \approx 0{,}45 \div 0{,}50$ for jets of air and water (in particular,

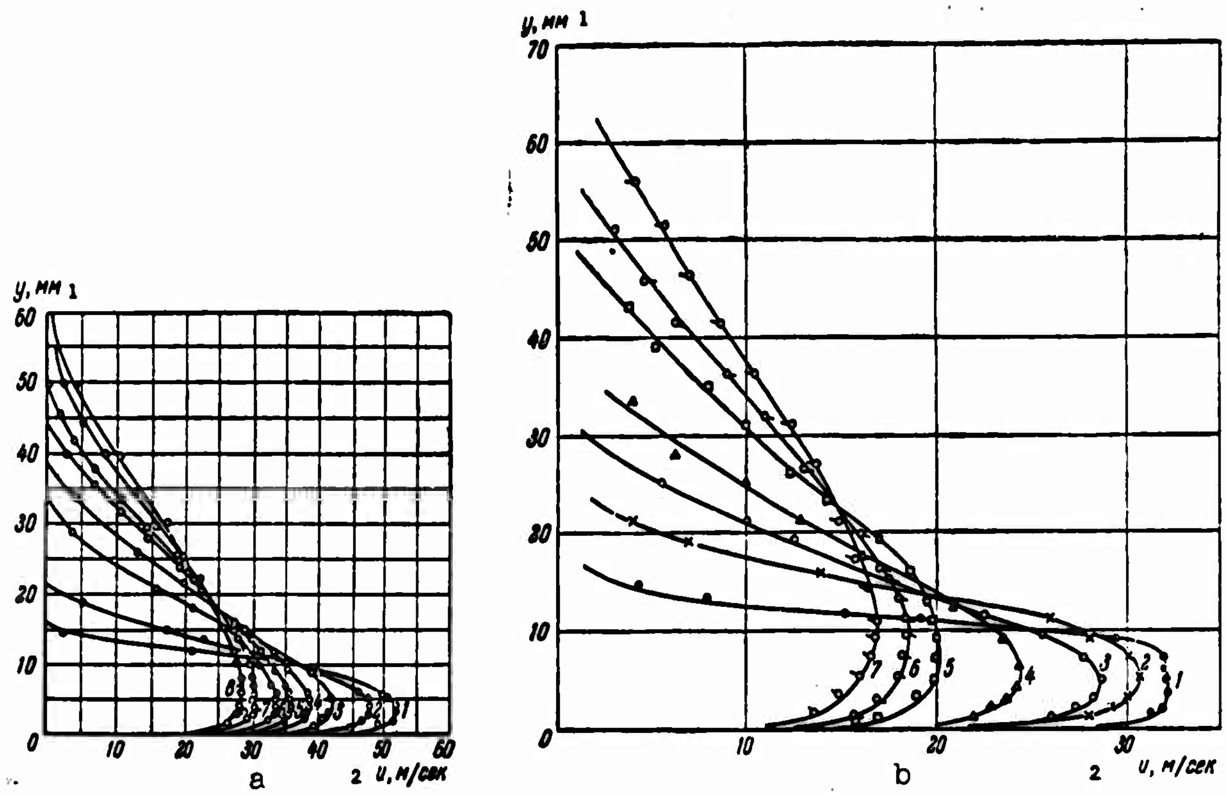

Fig. 12.5. Velocity distributions in turbulent semilimited jets of air (a) and water (b) (after data from [66]). 1) mm; 2) m/sec.

a

№	1	2	3	4	5	6	7	8
x/d	4	8	12	16	20	24	29	34

b

№	1	2	3	4	5	6	7
x/d	2	6	12	15	22	28	34

F. Förtmann [244] and A. Sigalla [297] gave $-\alpha = 0{,}5$). The value calculated for a laminar jet is $-\alpha = 0{,}5$.

It is worth noting that differences in the flow conditions in the zone adjacent to the wall exert practically no influence (within the limits of measuring accuracy) on the form of the universal velocity distribution. The latter are in all cases virtually coincident (see Figs. 12.5 and 12.6). More than that, the very same universal distribution also applies with sufficient accuracy to the cross sections of a fan-type semilimited jet.

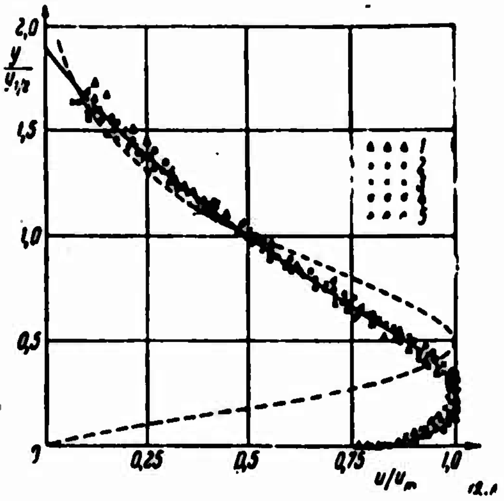

Fig. 12.6. Universal velocity distribution in a turbulent plane semilimited jet [66].

1) $u_0 = 44$ m/sec
2) $u_0 = 52$ m/sec — Air jet;
3) $u_0 = 13.8$ m/sec
4) $u_0 = 2.7$
5) $u_0 = 3.2$ — water jet;

– – – – velocity profile calculated for a laminar jet.

12.3. CALCULATION OF A SEMILIMITED JET

The experimental values of the constants of self-similarity given in the preceding section are close to those which are obtained from dimension considerations or direct calculations for self-similar jet, under the assumption of $v_T = bu_m$. In this case, however, the calculated velocity distribution in a jet cross section $\frac{u}{u_m} = F'(\varphi)$ should have been coincident with the laminar one shown in Table 7.1. This distribution (dashed line in Fig. 12.6) resembles only qualitatively the experimental one. When we compare the maxima of the curves, we see that the outer parts of them coincide virtually; they also agree with the distribution for a plane free turbulent jet (in the coordinates $\frac{u}{u_m} = F(\frac{y}{y_{1/2}})$) for the outer part.

The inner part of the distributions, both the experimental and the calculated ones for $v_T = bu_m$, and even the position of the velocity maximum, differ essentially. This difference, which has almost no influence on the self-similarity constants of the jet, can be explained (as also the shape of the function $K(x)$) by the structure of the boundary layer adjacent to the wall. NOte that it has already been indicated in F. Förtmann's first paper [244] that a calculation of the velocity distribution in a turbulent semilimited jet must be built up by way of "joining" the various solutions for the outer and inner zones of the jet's boundary layer. As already mentioned, the solution for the free turbulent jet applies to the outer zone; for the inner zone an exponential distribution is recommended in paper [244] and, with particular detail, in paper [246], which is typical for a turbulent boundary layer, for example, "the law of one seventh" for the Blasius zone. An analogous calculation is contained in the monograph [11].

These calculations are based on correct representations of the initial conditions with respect to the different structures of the inner and outer parts of the turbulent semilimited jet. As regards the "law of one seventh," it applies approximately to flows in the turbulent boundary layer adjacent to the wall. In the case of a laminar wall layer, the inner part of the distribution is of course in no connection with Blasius' law of resistance. In this case, the qualitative features of the anisotropy reminds us of the well-known problem of the laminar flow in an annular channel with a characteristic asymmetry of the logarithmic velocity distribution (a displacement of the maximum to the inner wall).

For an evolved turbulent flow in all zones of a plane jet a relatively simple approximation method (already mentioned above) can be developed on the basis of the following suppositions. We assume first-

ly that the constants of self-similarity are $\alpha = -\frac{2}{3}$, $\beta = -1$ and suitable to describe the entire flow. Secondly, and this is the main point, we develop an interpolation formula for the coefficient of turbulent viscosity. For this purpose, we assume that near the wall, according to Prandtl, $\nu_T \sim l^2 \frac{\partial u}{\partial y} \sim y^2$, since $l \sim y$ and $\frac{\partial u}{\partial y} \approx \text{const}$ (the velocity distribution is almost linear). Away from the wall, in the inner part of the jet, $\nu_T = bu_m \sim x^{1/4}$. For the layer as a whole we put

$$\nu_T = f(\varphi)\, bu_m,$$

where $f(\varphi)$ is the interpolation function which, in particular, may be taken as equal to $f(\varphi) = 1 - e^{-C\varphi^2}$; the coefficient C plays here the part of the second empirical constant.

The self-similar differential equation will under these suppositions take the form

$$[f(\varphi)\, F''(\varphi)]' + F(\varphi)\, F''(\varphi) + 2F'^2(\varphi) = 0$$

with the boundary conditions

$$F = F' = 0 \quad \text{if} \quad \varphi = 0,$$
$$F' = 0 \quad \text{if} \quad \varphi = \infty.$$

The numerical solution of this equation with the above form of the interpolation function $f(\varphi)$ and a value of the constant $C \approx 30$ agrees satisfactorily with experiments for an evolved turbulent flow [66]. To this solution corresponds the solid line in Fig. 12.6. In a wider range of experimental data, the value of the constant C would in fact prove to be a function of the Reynolds at present, however, such a precise definition would be premature as we have no reliable experimental data.

Other variants of calculation suggested in paper [246] or [11] are not discussed here because of their complexity and the preferableness of the continuous interpolation procedure.

Finally we want to remark that just this interpolation method was used successfully by V.R. Aubarikova and A.T. Trofimenko [24] for the solution of the thermal problem (with the boundary conditions $\frac{\partial T}{\partial y} = 0$ at $y = 0$) of a semilimited turbulent jet. Data on the heat transfer in semilimited turbulent jets are also contained in paper [143] and others.

As regards the calculation of a turbulent fan-type semilimited jet, which in particular is interesting as a model of motion arising in the efflux of a round jet streaming perpendicularly against an obstacle, an analogous interpolation procedure can be applied to it. In this case, $\beta = -1$ and, in a first approximation, $\alpha = -4/3$. This value corresponds to the supposition that $v_T = bu_m$. Note that in papers [296, 297] the experimental values $\alpha = -1.12$ and $\beta = -0.94$ are given.

In any case, for a turbulent fan-type semilimited jet $|\alpha| > 1$, approximately $\alpha \approx -(1{,}2 \div 1{,}3)$ (in contrast to $\alpha = -2/3$ for a laminar jet).

Detailed investigations of this type of a turbulent jet flow are indubitably interesting for both theory and practice.

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11, 15, 16, 17, 18, 19, 24, 66, 68, 114, 143, 191, 192, 193, 226, 244, 246, 296, 297.

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280	See also paper [296] in which $\alpha \simeq -0.55$, etc.
283	The distribution curve is taken from the monograph [130].

Chapter 13

SELFSIMILAR TURBULENT GAS JETS

13.1. ON THE MECHANISM OF TURBULENT MIXING IN GASES

The extension of the ideas and methods of solving selfsimilar problems of the theory of turbulent jets of incompressible fluid to analogous flows of compressible gas requires preliminary discussions of, at least, two problems. We are here concerned with the qualitative representation of the process of turbulent mixing in gases and the problem for which flow characteristics the presence of universal distributions (in the cross sections of self-similar gas jets) may be assumed with high probability.

Though they are of different generality, both problems are linked with each other and with the choice of the mathematical characteristics which determine the evolution of turbulent jets of compressible gas.

Recall that the turbulent mixing of an incompressible viscous fluid is connected with an instability of the interface of parallel streams. An illustrative idea on the mechanism of the appearance of turbulent mixing in the interface of such flows can be obtained when, following L. Prandtl, we apply the Bernoulli equation to the two neighboring fluid streams. Let us assume that in one of them a local random extension occurs. It will be accompanied by a decrease of velocity and an increase in pressure in this stria (in the perturbed cross section). At the same time, in the same section of the neighboring stria, a con-

striction occurs, the velocity of motion in it rises and the pressure drops. Perpendicularly to the interface of the striae a force will arise, namely the pressure gradient, which favors a further growth of the perturbed interface. As a result, with appropriate relations between the forces of inertia and the forces of viscosity (preventing perturbations), the random buckling will be amplified until a "neck" forms which may then tear and some fluid may pass over from one stria to another. These transitions (in agreement with the continuity equation) will occur in the one and the opposite direction. As a result, the interface becomes broader, thus forming the zone of turbulent mixing.

These qualitative considerations have been verified qualitatively in a well-known theorem of hydromechanics, the theorem of the absolute instability of a tangential discontinuity [122].

Let us try to transduce this procedure to the flow of two adjacent striae of compressible gas. Note that also in the case of $\sigma =$ $=$ const in the Bernoulli equation for an incompressible fluid a pressure variation is connected immediately not with the velocity (the kinematic characteristic of the flow) but with the value of the dynamic pressure, ρu^2.

In a compressible gas a change in the cross section of a stream tube causes a change of opposite sign of the flux density ρu (mass velocity) of the fluid. Since the relative velocity of motion of a gas in the two mixing elementary striae will be subsonic, the signs of variation of linear velocity, mass velocity and dynamic pressure (momentum flux density ρu^2) agree when the stria widens, i.e., all these quantities decrease while the pressure increases. In contrast to this, in the zone of constriction of a stria the pressure drops and the values of u, ρu, and ρu^2 grow. In other words, the qualitative picture of the stability losses of the interface remains the same as in the case

of the incompressible fluid: in both cases the appearance of a transverse pressure gradient which disturbs the stability of the interface is connected with a difference in the values of the quantity ρu^2 [50].

Applying this to a gas jet discharged into a space filled by a gas of another density, in the general case, where the values of the velocity and the density of the gases in the jet and the medium surrounding it (in a parallel stream) are different, we may suppose that turbulent mixing will occur the more intense the greater the difference between the local values of ρu^2, and not of u or ρ individually.

In a first approximation also the opposite conclusion must hold true, namely that there is no intense turbulent exchange (ignoring the isotropic turbulence and, of course, the molecular mixing) when the values of ρu^2 are the same in the two mixing parallel flows, in spite of different values of velocity and density separately*).

The assumption of the decisive role of ρu^2 in the turbulent mixing of a gas is faced by another widespread point of view [11, 217, 218], according to which in all cases, i.e., in both an incompressible fluid and a gas, turbulent mixing is chiefly caused by differences of the values of the mean velocities. As regards the difference in the densities, it may be given the part of an additional factor in this scheme, a factor which intensifies the turbulent exchange.

The conclusions which can be drawn from these two conceptions under definite conditions are exactly opposite. Let us imagine that two parallel flows of gas of the same velocity but different densities, and thus with different values of ρu^2, produce a zone of turbulent mixing. Its existence is due to the density difference. Let us now assume that, without changing the density values, a difference in the velocity values is created additionally, which is [illegible] that the values of ρu^2

which is the main cause of the instability, the mixing intensity must be diminished. If it is, however, the difference in the velocities which is the principal cause of turbulent exchange, the mixing intensity must increase.

In this thought experiment, one and the same effect yields quite opposite results in the two schemes considered.

In spite of the fact that we know of a considerable number of experimental observations which, for the most part, speak in favor of ρu^2 playing the decisive part in processes of turbulent mixing of gases [114,193], it was necessary to carry out special tests which should yield more complete proofs of the validity of this assumption.

Such experiments were conducted by I.B. Palatnik on the expansion of a jet of hot gas in a colder parallel flow [64] with three values of the ratio between the gas temperature T_0 in the jet (combustion products) and the air temperature T_∞ in the wake. Figure 13.1 shows the experimental curves of the dependence of the quantity $\Delta T_m = T_m - T_\infty$ divided by its initial value $\Delta T_0 = T_0 - T_\infty$ for an axisymmetric jet, with one and the same value of $T_0/T_\infty \approx 3$. As we see from the figure, the curves $\frac{\Delta T_m}{\Delta T_0}$ for three different nozzle distances, possess a clearly marked maximum in the range of a value of $\frac{(\rho u^2)_\infty}{(\rho u^2)_0} \approx 1$. The value of $\frac{\Delta T_m}{\Delta T_0}$ characterizes the mixing intensity; the more $\frac{\Delta T_m}{\Delta T_0}$ differs from the initial value (which is equal to unity), the more intense is the mixing. On the other hand, the maximum of the value of $\frac{\Delta T_m}{\Delta T_0}$ corresponds to minimum mixing. Under conditions where $\frac{u_0^2}{u_\infty^2} \approx 1$, and the ratio $\frac{(\rho u^2)_\infty}{(\rho u^2)_0} = \frac{\rho_\infty}{\rho_0}$ being essentially different from unity, the value of $\frac{\Delta T_m}{\Delta T_0}$ drops rapidly. If we go from this value (righthand side of Fig. 13.1) along the curve toward the origin of coordinates, the value of $\frac{\Delta T_m}{\Delta T_0}$ will grow and the intensity of turbulent mixing will drop until a ratio of

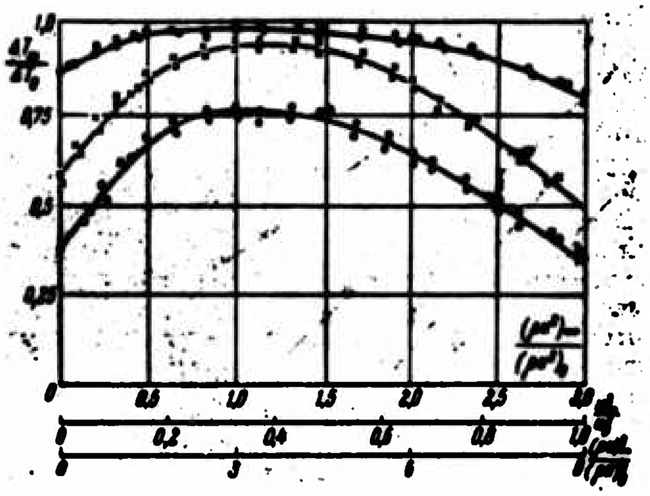

Fig. 13.1. Mixing of parallel gas jets of different densities (with $T_\infty/T_0 = 3$ [illegible]. ○○○ $x/d = 2.5$; ××× $x/d = 5.0$; ●●● $x/d = 7.5$.

$\frac{(\rho u^2)_\infty}{(\rho u^2)_0} \approx 1$. is reached. As mentioned above, if differences in the velocities were the principal cause of turbulent exchange, the value of $\frac{\Delta T_m}{\Delta T_0}$ must drop and the mixing intensity must rise (when moving in the same way along the curves). This, however, is contradictory to the experiments.

With a further moving along the curves toward the origin (in Fig. 13.1) the quantity $\frac{\Delta T_m}{\Delta T_0}$ decreases, which is in agreement with the assumption of ρu^2 playing the main part (but without experimental data for the right half, the figure might also be explained by an increasing influence of the velocity differences).

The experimental results discussed (partly represented in Fig. 13.1) justify the assumption that under these conditions the differences in the values of momentum flux density ρu^2 play the principal part and we make use of this in practice, when calculating the evolved turbulent exchange (neglecting effects of molecular nature).

This conclusion cannot be considered an exhaustive answer without

a detailed investigation of the turbulent structure of the flow and an explanation of the possible connections between the characteristics of turbulence (intensity of pulsation, dimensions of the vortices at the interface, the characteristic frequencies, etc.) and the physical properties of the mixing gases whose influence may be considerable at relatively small Reynolds numbers in the initial region of the flow. In any case,the limits of its applicability must be established according to experiments. As regards the nature of the relatively intense mixing of flows possessing the same velocities, for its explanation it is essential that the qualitative image of turbulent mixing considered at the beginning of this section refers not to the average but to the non-stationary actual motion. A transition to a coordinate system moving with the average velocity (which is the same for both flows) will therefore not remove the turbulent exchange as can be seen at the first sight. In fact, even with equal mean values of the velocity, the difference of the actual values of ρu^2 is maintained which would not exist in a system at rest (for example, in a container filled with two parallel layers of gases of different density which are at rest while the container may be either at rest or performing an inertial movement).

It is at present one of the main tasks in this field to develop effective approximation methods of calculation whose accuracies must correspond to the technical and laboratory experimental data on investigations of jets. The mathematical methods chosen must, of course, tend to agree with the physical representations and the results of physical experiments*) as well as possible, apart from the requirement for their simplicity, illustrativeness and the like. A detailed analysis of the influences of the various factors must be carried out by way of experiment as they are often concealed by the considerable

spread of the experimental data.

These considerations, which refer to the present level of investigations of the turbulent motion of a compressible gas, indicate that extreme care is necessary when the methods of calculation, individual approximations and the like are extrapolated beyond the limits verified by experiment.

13.2 THE UNIVERSAL DISTRIBUTIONS OF ρu^2 AND OTHER QUANTITIES

Let us now turn to the second of the problems stated in the previous section. We shall try to find out which of the characteristics of a turbulent flow must describe the properties of selfsimilar jet flows of compressible gas. For example, in the case of a turbulent gas source-jet, i.e., the motion at a great distance from the nozzle, with a noticeably variable density distribution and, consequently, such one of the absolute temperature, we shall investigate whether, as in the case of the turbulent flow of an incompressible fluid, the velocity distributions in the jet cross sections can be assumed universal.

Recall that no characteristic dimension exists for a turbulent motion produced by a source-jet. Therefore, as in the case of ρ = const, the nondimensional coordinate describing the self-similar motion may only be the ratio y/x.

Let us briefly discuss the result of an attempt of developing nondimensional distributions of the characteristics of a compressible gas jet under the assumption of similar velocity distributions $\frac{u}{u_m}=f_1\left(\frac{y}{x}\right)$. From the conditions of momentum flux conservation

$$J_x=\int \rho u^2\,ds=\text{const},$$
$$ds=(2\pi y)^k\,dy \quad (k=0;\,1)$$

it follows that besides the relative velocity u/u_m also the relative density distribution $\frac{\rho}{\rho_m}=f_2\left(\frac{y}{x}\right)$ must be universal and, with constant pressure, also the absolute temperature distribution $\frac{T}{T_m}=f_3\left(\frac{y}{x}\right)$. This,

however, indicates that the distribution of the surplus temperature

$$\frac{\Delta T}{\Delta T_m}=\frac{T-T_\infty}{T_m-T_\infty}=\frac{\frac{T}{T_m}-\frac{T_\infty}{T_m}}{1-\frac{T_\infty}{T_m}}$$

cannot be universal, as the ratio T_∞/T_m depends on the single coordinate x.

Considering the second integral invariant, namely the condition of conservation of the flux of surplus heat content $Q=\int_S \rho c_p u\,\Delta T\,ds$, we see that the condition Q = const, cannot be fulfilled if the surplus temperature $\frac{\Delta T}{\Delta T_m}$ is not a universal function of y/x (as velocity and density are).

Thus, on the one hand, in order to satisfy the conditions J_x = = const and Q = const it is necessary that the relative values of the integrands are universal functions of the variable y/x while on the other hand all the three quantities (u, T and ΔT) cannot possess universal distributions at the same time.

All these contradictions are avoided if, following the papers [50 etc.], one assumes that not the velocity and temperature distributions are universal separately but the distribution of the momentum flux density ρu^2 and also that of the flux density of the surplus heat content $\rho u c_p \Delta T$ etc.

In fact, when we put $\frac{\rho u^2}{(\rho u^2)_m}=f\left(\frac{y}{x}\right)$, we satisfy immediately the condition J_x = const and obtain in addition to this the law of decline at the jet axis: $(\rho u^2)_m\sim x^{-(k+1)}\sim x^{2\alpha}$, where α = −1 in an axisymmetric source-jet and α = −1/2 in a plane one. We then also obtain $\frac{\rho u c_p \Delta T}{(\rho u c_p \Delta T)_m}=F\left(\frac{y}{x}\right)$. Here the condition Q = const is satisfied and, moreover, the law of variation of $(\rho u c_p \Delta T)_m$ along the axis of the jet is obtained: in an axisymmetric jet this quantity is proportional to x^{-2} and in a plane one to $-x^{-1}$.

The supposition on the similarity of the distributions of the momentum flux density and the surplus heat content enable us to avoid the difficulties with the derivation of universal functions for self-similar jet motions of compressible gas. If we assume that the turbulent exchange in a compressible gas is due to a difference in momentum flux density ρu^2, and not in ρ or u separately, it is quite natural to suppose that the distribution of this quantity ρu^2 in a self-similar turbulent gas jet will not depend on the initial efflux conditions, that is, it will be universal. The very same will also hold good for the other "substantials": the flux density of surplus heat content, of the surplus total heat content (calculated from the stagnation temperature for high velocities of motion), the surplus mass flux in the diffusion problem, and the like. It is precisely these quantities which, besides ρu^2, enter the integral conditions of conservation.

Let us now discuss it from the physical point of view in which conclusions on the laws of expansion of turbulent gas jets the supposition of the universality of the distributions of ρu^2, etc. will result.

From the condition $(\rho u^2)_m \sim x^{2a}$ it follows that in the case of the expansion of a gas of lower density in a denser atmosphere, ith otherwise unchanged conditions, the velocity will decrease more rapidly along the axis, and in the opposite case (efflux of denser gas into a medium of lower density) more slowly than when the densities are equal*).

This conclusion applies to the velocity distributions in the jet cross-sections which result immediately from the universality of the distribution of $\frac{\rho u^2}{(\rho u^2)_m}$.

Analogous conclusions are also obtained with respect to the distributions of surplus temperature along the axis of the jet and also in its cross-sections.

All these effects which are connected with the compressibility

in turbulent gas jets are qualitatively the same as in laminar jets (see Fig. 8.2).

The supposition on the similarity of the distributions of ρu^2, etc. in self-similar turbulent jet flows of gas, continued logically, results in the identity of the universal distributions, e.g., $\frac{\rho u^2}{(\rho u^2)_m} = f(\varphi)$, where $\varphi = \frac{y}{ax}$, for both gas and incompressible fluid ($\frac{u^2}{u_m^2} = f(\varphi)$ in the particular case of a jet with ρ = const). This supposition may be extended to all mathematical laws for turbulent exchange in gas jets. This results in the possibility of using data referring to the well-known self-similar turbulent jets of incompressible fluid in order to calculate analogous gas jets. For this purpose, it is first necessary (case ρ = const) to represent the momentum flux density, that of the heat content, etc., as functions of the reduced coordinates.

The question which remains open in this connection is that for the intensity of variation of the experimental constant a of the "jet structure," which enters the universal variable $\varphi = \frac{y}{ax}$, in the transition from incompressible fluid to gas. It is not clear, in particular, whether the quantity a, under otherwise equal conditions, depends explicitly on the density ratio of the gas in the jet and that in the surrounding space, etc. Recall in this connection that the constant a has no universal value which is conserved for a definite type of jet and with ρ = const. Its value is determined by the influence of various factors which are difficult to take into account, macroscopic (shape of nozzle, nonuniformity of initial distributions, etc.) and conditionally microscopic characteristics (intensity of turbulence, initial ratio of the pulsations of the longitudinal and the transverse velocity components, the scales, etc.) and, finally, of physical parameters as the ratio of the densities, viscosities and the like, of the jet and the surrounding medium. It is obvious that only an analysis of reliable experiment-

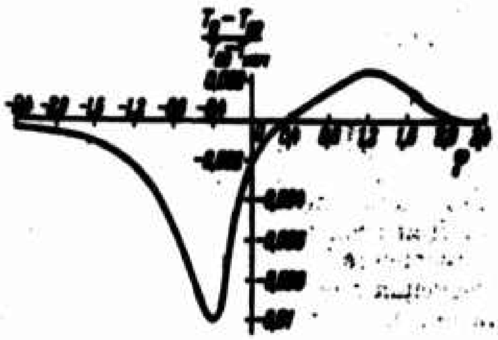

Fig. 13.2. Distribution of stagnation temperature for the boundary of a turbulent jet.

al data permits a progress in this complex problem. The simple solution, namely to choose *a priori* the value of one of the composite constants and its dependence on the density ratio, cannot be unambiguous in the wide range of experimental conditions. Both considerations allow the conclusion that in the case of a high level of initial turbulence the difference between the a-values for gas jets of different densities will be very small, with a medium level it will be noticeable and if the initial level of turbulence is low, the a-values will differ considerably. A more precise quantitative formulation of this conclusion will follow when we compare calculation and experiment.

It is essential that, independently of the influence of the parameters of compressibility on the value of the constant a (directly or indirectly, e.g., through the elevation of the level of turbulence in the process of heating, combustion, etc.), an appropriate choice of this value as a rule results in a better agreement of the universal laws (the distributions of ρu^2 etc.) with the experiments and with one another, for different density values.

Among the physical laws of turbulent transfer in gas jets, one

should also mention the effect of local redistribution of total energy, which has already been discussed in connection with the laminar flow. In all investigations known to the authors the so-called turbulent Prandtl number for a jet motion (in practice also in other cases) is smaller than unity. Hence it follows that in an integral-adiabatic gas jet, that is, in the case of efflux of gas where the temperatures in the tank and the surrounding medium are the same, the stagnation temperature distributions in the jet cross-sections (and also along the jet) cannot be constant. The energy in the faster striae of the gas must increase at the expense of energy losses of the slower striae, etc.

Calculations of this kind carried out under various assumptions on the velocity and temperature distributions [47, 48] verified the presence of a local redistribution of total enthalpy in turbulent gas jets which at present is universally recognized.

In this connection arises the necessity of verifying also in this respect the calculation on the scheme of similarity of ρu^2. Without entering into details, we restrict ourselves to indicating that this way of calculation [61] and [145, 146] showed that also in this respect the method of similarity of ρu^2 yields physically correct results. By way of example, we have shown in Fig. 13.2 the distribution of total enthalpy in the boundary layer at the edge of a turbulent integral-adiabatic plane jet. Considered from the quantitative point of view, the effect of redistribution is hardly essential for the dynamic problem, for the thermal problem, however, it may be important in principle to take it into account.

13.3. CONSIDERATIONS OF DIMENSIONALITY

Let us now discuss the nature of the basic laws governing the self-similar turbulent jet flows of compressible gases. For this purpose, as before, we make use of dimension considerations which we apply to the

same concrete cases of motion in the mixing zone of a uniform flow with nonmoving gas, and the expansion of source-jets (plane and axisymmetric). In each of these cases, using the initial given quantities we express the characteristic quantities as functions of the given parameters and the coordinate x, accurately, except for a certain nondimensional function $f(\varphi) = f\left(\frac{y}{ax}\right)$, which includes numerical factors.

To begin with, we consider the problem of the edge of a plane-parallel jet. In this problem, there are two initial quantities: the momentum flux density in the nonperturbed incoming flow, $(\rho u^2)_0$, and the gas density in the quiescent surrounding medium, ρ_∞, which, as already mentioned, are completed by the coordinate x.

Taking the dimensions of the given quantities (ρu^2, ρ, x, etc.) into account and allowing the conventional sign $\sim$ to comprise const $f(\varphi)$, we can formulate the expressions of turbulent friction

$$\tau_T \sim \rho u^2 = \text{const}\, f_1(\varphi)\, \rho u^2,$$

of turbulent viscosity

$$\mu_T \sim \sqrt{\rho_\infty (\rho u^2)_0}\, x,$$

and of turbulent kinematic viscosity

$$\nu_T \sim \sqrt{\frac{(\rho u^2)_0}{\rho_\infty}}\, x.$$

If we take to the given quantities also the flux density of surplus heat content (and in the case of high velocity of motion, the total heat content $c_p T_0 = c_p T + \frac{u^2}{2}$), we obtain from the same considerations: the heat flux density

$$q = -\lambda_T \frac{\partial T}{\partial y} \sim \rho u c_p \Delta T = \text{const}\, f_2(\varphi)\, \rho u c_p \Delta T,$$

the turbulent thermal diffusivity

$$a_T \sim \sqrt{\rho_\infty^{-1} (\rho u^2)_0}\, x,$$

and the turbulent thermal conductivity

$$\lambda_T \sim \rho_\infty c_{p\infty} a_T.$$

With this, the distributions of the momentum flux density and the heat content are determined by universal functions of the coordinate φ :

$$\frac{\rho u^2}{(\rho u^2)_m} = F_1(\varphi) \text{ and } \frac{\rho u c_p \Delta T}{(\rho u c_p \Delta T)_m} = F_2(\varphi).$$

When we, as before, introduce a cross-sectional element $ds \doteq (2\pi y)^k dy$, for the source jets, we can unite the axisymmetric ($k = 1$) and the plane ($k = 0$) source jets in a single scheme. The given quantities are now the quantity J_x, the gas density ρ_∞ in the nonperturbed medium and, for the thermal problem, the flux density of surplus heat content $Q = \int\limits_S \rho u c_p \Delta T ds$.

Using these quantities and also the coordinate x, we obtain for the turbulent friction

$$\tau_T \sim J_x x^{-(k+1)},$$

the turbulent viscosity

$$\mu_T \sim \sqrt{J_x \rho_\infty x^{1-k}},$$

the turbulent kinematic viscosity and the thermal diffusivity

$$\nu_T \sim a_T \sim \sqrt{\frac{J_x}{\rho_\infty} x^{1-k}},$$

the turbulent thermal conductivity

$$\lambda_T \sim c_{p\infty} \sqrt{\rho_\infty J_x x^{1-k}},$$

and the flow rate of gas

$$G = \int\limits_S \rho u \, ds \sim \sqrt{\rho_\infty J_x x^{1+k}}.$$

The distributions of $\frac{\rho u^2}{(\rho u^2)_m}$ and $\frac{\rho u c_p \Delta T}{(\rho u c_p \Delta T)_m}$ in the jet cross-sections will be universal functions of the variable φ, where

$$(\rho u^2)_m \sim J_x x^{-(k+1)} \text{ and } (\rho u c_p \Delta T)_m \sim Q x^{-(k+1)}.$$

These formulas are very general. In particular, the well-known Prandtl formulas for turbulent friction in an incompressible fluid agree with those given here. In fact, with ρ = const, $l = cx$ and $\frac{\partial u}{\partial y} \sim \frac{u_m}{x} F(\varphi)$ the formula $\tau_T = \rho l^2 \left(\frac{\partial u}{\partial y}\right)^2$ for the round jet $\left(u_m \sim \frac{1}{x}\right)$, yields as above

$$\tau_T = \text{const}\, F^2(\varphi) \frac{1}{x^2},$$

while

$$\nu_T = \text{const}.$$

It should also be noted that the expression for the flow rate of the gas in the jet $G \sim \sqrt{\rho_\infty J_x x^{k+1}}$ can be rewritten in the form

$$\frac{G}{G_0} = \sqrt{\frac{\rho_\infty}{\rho_0}} \left(\frac{x}{d_0}\right)^{\frac{k+1}{2}},$$

if we introduce the conventional for the initial section of the equivalent gas jet of finite dimensions, namely, ρ_0 and u_0 and $s_0 = \left(\frac{\pi}{4} d_0\right)^k d_0$, where d_0 is the characteristic dimension of the outlet cross-section of the nozzle, $G_0 = \rho_0 u_0 s_0$ and $J_x = \rho_0 u_0^2 s_0$.

The expression for the rllative flow rate of gas in the jet, which holds true at a considerable distance away from the source, shows that the mass of gas added per second from the surrounding medium to the jet will grow in a linear proportion to the distance in an axisymmetric jet and proportional to the square root of the relative distance x/d_0 in a plane jet. In both cases the coefficient of proportionality in the formula for G/G_0 is equal to the square root of the density ratio ρ_∞/ρ_0 This indicates that the relative "ejecting capacity" of the gas jet is the higher the lower the gas density in the jet compared to the gas density in the surrounding medium.

In the outstanding paper by F.P. Ricou and D.B. Spalding [285] experimental data are given which have been obtained in direct measurements of the mass adjoined. As we see from Fig. 13.3 (for a round jet) the ex-

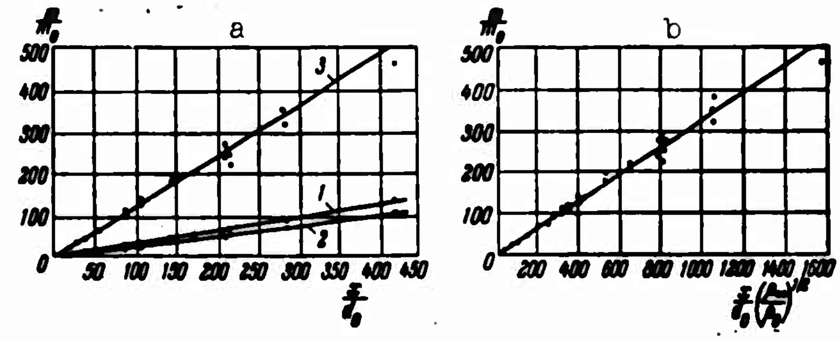

Fig. 13.3. Dependence of the relative rate of gas flow in the jet on the distance and the density ratio for the isothermal efflux of various gases into air (after data by F.P. Ricou and D.B. Spalding [285]. a) 1 — Efflux of air $(G/G_0 = 0{,}32\, x/d_0)$; 2 — efflux of propane and carbon dioxide $(G/G_0 = 0{,}26\, x/d_0)$; 3 — efflux of hydrogen $(G/G_0 = 1{,}2\, x/d_0)$; b) universal curve $\left(G/G_0 = 0{,}32\, (x/d_0) \sqrt{\frac{\rho_\infty}{\rho_0}}\right)$

periment verifies the dependence derived. The results of paper [285] result in a formula for the rate of gas flow in the form

$$\frac{G}{G_c} = 0{,}32 \frac{x}{d_0} \sqrt{\frac{\rho_\infty}{\rho_0}}.$$

The expressions we obtained here for the particular case of the motion of an incompressible fluid agree of course with the formulas given previously.

Let us now try to use the expressions derived in order to explain the influence of the parameter of compressibility, $\omega = \frac{\rho_\infty}{\rho}$ on the non-universal) velocity and temperature distributions. This is particularly easy for the distributions in the jet cross-sections.

In this case $\omega_m = \frac{\rho_\infty}{\rho_m}$ (the density on the axis of the source jet is taken as the characteristic density value of the jet). For the edge of the jet, it is obvious that $\omega = \omega_0 = \frac{\rho_\infty}{\rho_0}$, where ρ_0 is the gas density in the incoming flow. The necessity of knowing also the ratio of the gas densities in the jet and the surrounding medium becomes obvious

from physical considerations. So we also assume the quantity u_0, the velocity of the incoming flow, given.

Finally, for the transition from universal laws holding true on the jet axis to expressions which determine the variations of velocity and surplus temperature along the axis we shall also assume that the initial values of density ρ_0 and velocity u_0 are given (for the equivalent jet of finite dimensions). It should be stressed in this connection that the idea of assuming ρ_0 and u_0 given results from the tendency to simplify the analysis of the influence of the compressibility of gas on the velocity and temperature distributions and we never required previously, when the general law were established.

From the similarity of the distributions of ρu^2 in the cross sections it follows that

$$\frac{\rho u^2}{(\rho u^2)_m} = \left(\frac{u^2}{u_m^2}\right)_{\rho=\mathrm{const}} = F_1(\varphi),$$

and hence

$$\frac{u^2}{u_m^2} = \frac{\rho_m}{\rho} F_1(\varphi) = \omega F_1(\varphi).$$

With φ given, the ratio u/u_m will be higher or smaller than the value of $\left(\frac{u}{u_m}\right)_{\rho=\mathrm{const}}$ in an incompressible fluid according to whether $\omega > 1$ (jet of denser gas, e.g., cooler) or $\omega < 1$ (e.g., hot jet, gas of lower density).

In the first case, the velocity distributions are displaced away from the ordinate axis in the second case they lie closer to it. The very same also applies to the distributions of $\frac{\Delta T}{\Delta T_m}$ in the cross sections, since from the universality of the distributions of $\frac{\rho u^2}{(\rho u^2)_m}$ and $\frac{\rho u c_p \Delta T}{(\rho u c_p \Delta T)_m}$ it follows that

$$\frac{\Delta T}{\Delta T_m} = \left(\frac{\Delta T}{\Delta T_m}\right)_{\rho=\mathrm{const}} \sqrt{\frac{\rho_m}{\rho}} = \sqrt{\frac{\rho_m}{\rho}} F_2(\varphi).$$

(for simplicity we give it for c_p = const).

In the case of the efflux of a gas jet with high velocity the distribution of the flux density of total heat content is universal. The conclusion as to the surplus temperature we obtained here will therefore remain in force for the distribution of the excessive stagnation temperature (in the general case, the surplus total enthalpy).

From the qualitative point of view the influence of high velocity (more accurately, the Mach number) is analogous to the influence of a higher gas density in the jet.

In order to determine the laws of variation of velocity and surplus temperature along the axis of a compressible gas jet we use the expressions

$$\frac{u_m^2}{u_0^2} = \frac{(\rho u^2)_m}{(\rho u^2)_0}\frac{\rho_0}{\rho_m} = \left(\frac{u_m^2}{u_0^2}\right)_{\rho=\mathrm{const}}\frac{\rho_0}{\rho_m}$$

and

$$\frac{\Delta T_m}{\Delta T_0} = \frac{(\rho u c_p \Delta T)_m}{(\rho u c_p \Delta T)_0}\frac{\rho_0 u_0}{\rho_m u_m} = \left(\frac{\Delta T_m}{\Delta T_0}\right)_{\rho=\mathrm{const}}\sqrt{\frac{\rho_0}{\rho_m}},$$

If a gas of lower density streams into a denser atmosphere the velocity drop on the jet axis occurs more rapidly than in the case of equal densities. On the other hand, a dense jet of gas is damped more slowly.

Thus, in all cases of self-similar flows of gas, both along the jet axis and in the cross sections, the decrease of velocity and surplus temperature occurs more intensely if the gas density in the jet is lower than in the surrounding medium, and less intense in the opposite case compared with the turbulent mixing in the case of equal densities. This indicates that a jet of denser gas, as it is more "inert," exists as such over a greater distance from the nozzle orifice. In the limiting case of $\rho = \infty$ the jet behaves like a solid rod moving in a medium

of low density; the velocity along the axis and in the whole cross section is the same as in the initial section. This result agrees completely with the qualitative scheme shown in Fig. 8.2 for laminar gas jets.

It must be borne in mind that a full-value comparison of the two cases of efflux of a gas into a space filled by a gas of equal or different density is a difficult task. As we see from precisely this Fig 8.2, a density which varies within the region of the flow changes both the longitudinal and the tran sverse velocity and temperature distributions of $\rho u^2, \rho u c_p \Delta T$ etc., are not subject to such a great influence of the compressibility parameters (together with other factors not taken into account). With an appropriate choice of the structural constant a the distribution of ρu^2 is as a rule universal. The problem of comparing the effective thicknesses (particularly in the initial part of the jet) determined according to the velocity distribution must therefore be approached with great care. This may in fact explain the lack of coordination in this problem, sometimes even in the determination of the nature of influence of compressibility on the basis of one and the same experimental data.

13.4. THE MATHEMATICAL PROCEDURE OF SIMILARITY OF ρu^2

The calculation of self-similar turbulent jet flows of compressible gas, just as such of incompressible fluid, may be carried out on the basis of various semiempirical mathematical procedures. Each of them is based on one or several assumptions which are usually extrapolated from the region of ρ = const. As examples we may mention the supposition of the similarity of the velocity distributions or Prandtl's modified law of the thickness increase of the mixing zone in the papers by G.N. Abramovich [7, 11, 12], the transition in the theory of the mixing length to A.A. Dorodnitsyn's variables, suggested in his time by V.Ya.

Borodachev and L.Ye. Kalikhman, the successive iteration of solutions on the basis of the method of small perturbations [25], etc.

For the agreement between the results of calculation and experiment, empirical constants are taken from the latter, owing to which the verification of the usefulness of the mathematical procedure becomes somewhat conditional. In fact, in those cases where the calculation results disagree with the physical conditions (for example, when the method of the mixing length is transferred to the plane of A.A. Dorodnitsyn's variables, etc.), the inapplicability of individual procedures becomes evident. In the absence of an obvious contradiction, the choice of the optimum mathematical procedure loses the unambiguousness and becomes somehow subjective.

From the viewpoint of practice, the calculation of self-similar turbulent gas jets can be simplified considerably if, as shown above, the expressions of the universal distributions of the quantities $\rho u^2, \rho u c_p \Delta T$ etc., as functions of the variable φ in the cross sections (and of the coordinate x along the jet) and of the integral characteristics J_x and Q, are assumed to be the same as in the general case of a variable density distribution and in the particular case of ρ = const.

Let us take this as an initial supposition. The procedure of the subsequent calculation may be illustrated by way of example of an axisymmetric source jet. To define the problem concretely, we consider the problem of the efflux of a hot (or cold) — relative to the surrounding medium — gas jet with Mach numbers M << 1. In order to simplify the formulation as much as possible and to approach the case ρ = const, we introduce new variables [50, 58]. We denote the fundamental variable by

$$U = \sqrt{\rho u^2} = \sqrt{\rho}\, u$$

and express the other physical quantities in terms of it:

$$U = \sqrt{\rho}\, u, \quad V = \frac{\rho u v}{U} = \sqrt{\rho}\, v,$$
$$J = \frac{\rho u c_p \Delta T}{U} = \sqrt{\rho}\, c_p \Delta T$$

(or $J_0 = \sqrt{\rho} c_p \Delta T_0$ for a motion at high velocity). Analogous variables must be introduced in the equations of turbulent diffusion in gas jets. Linking the new variables U, V, J, etc. with the old ones u, v, $i = \rho u c_p \Delta T$, we arrive at the formulas used in the previous section:

$$\frac{u}{u_m} = \sqrt{\frac{\rho_m}{\rho}}\frac{U}{U_m} = \sqrt{\frac{\rho_m}{\rho}}\frac{F'(\varphi)}{\varphi}, \text{ where } \frac{F'(\varphi)}{\varphi} = \left(\frac{u}{u_m}\right)_{\rho=\text{const}},$$
$$\frac{\Delta T}{\Delta T_m} = \sqrt{\frac{\rho_m}{\rho}}\frac{J}{J_m} = \sqrt{\frac{\rho_m}{\rho}}\,\theta(\varphi), \text{ where } \theta(\varphi) = \left(\frac{\Delta T}{\Delta T_m}\right)_{\rho=\text{const}},$$

or

$$\frac{\Delta T_0}{\Delta T_{0m}} = \frac{T_0 - T_\infty}{T_{0m} - T_\infty} = \sqrt{\frac{\rho_m}{\rho}}\left(\frac{\Delta T}{\Delta T_m}\right)_{\rho=\text{const}}$$

etc., for the distributions in transverse cross sections, and obtain analogous formulas for the variation of u_m/u_0 etc., along the jet axis.

For these calculations it is necessary to determine the dependence of ρ_m/ρ on the variable φ (or ρ_0/ρ_m for the jet axis on the coordinate x) whereafter all results in an algebraic reduction. For this, let us consider, e.g., the formula

$$\frac{\Delta T}{\Delta T_m} = \sqrt{\frac{\rho_m}{\rho}}\,\theta(\varphi),$$

in which we substitute

$$\frac{\Delta T}{\Delta T_m} = \frac{T - T_\infty}{T_m - T_\infty} = \frac{\frac{T}{T_m} - \frac{T_\infty}{T_m}}{1 - \frac{T_\infty}{T_m}} = \frac{\frac{\rho_m}{\rho} - \frac{\rho_m}{\rho_\infty}}{1 - \frac{\rho_m}{\rho_\infty}}.$$

The ratio $\frac{\rho_m}{\rho_\infty}$ will be considered to be a given parameter of the problem. The equation

$$\frac{\rho_m}{\rho} - \frac{\rho_m}{\rho_\infty} = \left(1 - \frac{\rho_m}{\rho_\infty}\right)\sqrt{\frac{\rho_m}{\rho}}\,\theta(\varphi)$$

after a transformation, will then take the form of a quadratic equation in the quantity $r=\sqrt{\frac{\rho_m}{\rho}}$:

$$r^2-\left(1-\frac{\rho_m}{\rho_\infty}\right)\theta(\varphi)\,r-\frac{\rho_m}{\rho_\infty}=0.$$

From this equation we obtain

$$r=\left(1-\frac{\rho_m}{\rho_\infty}\right)\frac{\theta}{2}\pm\sqrt{\left(1-\frac{\rho_m}{\rho_\infty}\right)^2\frac{\theta^2}{4}+\frac{\rho_m}{\rho_\infty}}=$$

$$=\left(1-\frac{\rho_m}{\rho_\infty}\right)\frac{\theta}{2}\left[1\pm\sqrt{1+\frac{4\frac{\rho_m}{\rho_\infty}}{\theta^2\left(1-\frac{\rho_m}{\rho_\infty}\right)^2}}\right].$$

The "plus" sign in front of the square root corresponds to the case of $\frac{\rho_m}{\rho_\infty}<1$, i.e., a case where the gas density on the jet axis is minimum (e.g., the efflux of a hot gas into a colder and denser medium), the "minus" sign corresponds to the case of $\frac{\rho_m}{\rho_\infty}>1$ the gas density on the axis is maximum (e.g., a jet of cold gas discharged into an atmosphere of less dense hot gas). Here, with $\frac{\rho_m}{\rho_\infty}<1$ also $r < 1$; with $\frac{\rho_m}{\rho_\infty}>1$ at the same time also $r > 1$.

In an analogous way, we also determine the variation of the ratio $\frac{\rho_0}{\rho_m}$ along the jet axis, if ρ_0 is the initial value of the gas density in the jet. Here to,, we obtain from the analogous quadratic equation $\frac{\rho_0}{\rho_m}>1$ with $\frac{\rho_0}{\rho_\infty}>1$ and, correspondingly, $\frac{\rho_0}{\rho_m}<1$ with $\frac{\rho_0}{\rho_\infty}<1$.

Somewhat more complex is the calculation in the case of a jet of high velocity. Let us illustrate this by way of example, considering the distribution of $\frac{\Delta T_0}{\Delta T_{0m}}$ in the cross section of a round jet. We have

$$\frac{u}{u_m}=\sqrt{\frac{\rho_m}{\rho}}\frac{U}{U_m}=\sqrt{\frac{\rho_m}{\rho}}\frac{F'(\varphi)}{\varphi},\quad \frac{\Delta T_0}{\Delta T_{0m}}=\sqrt{\frac{\rho_m}{\rho}}\frac{\Delta T}{\Delta T_m}=\sqrt{\frac{\rho_m}{\rho}}\theta(\varphi),$$

and since

$$\frac{\Delta T_e}{\Delta T_{em}} = \frac{T+\frac{u^2}{2c_p}-T_\infty}{T_m+\frac{u_m^2}{2c_p}-T_\infty} = \frac{\frac{T}{T_m}+\frac{u^2}{u_m^2}\frac{u_m^2}{2c_pT_m}-\frac{T_\infty}{T_m}}{1+\frac{u_m^2}{2c_pT_m}-\frac{T_\infty}{T_m}} =$$

$$= \frac{\frac{\rho_m}{\rho}+\frac{\varkappa-1}{2}M_m^2\frac{u^2}{u_m^2}-\frac{\rho_m}{\rho_\infty}}{1+\frac{\varkappa-1}{2}M_m^2-\frac{\rho_m}{\rho_\infty}},$$

We finally arrive once again at a quadratic equation

$$\frac{\rho_m}{\rho}+\frac{\varkappa-1}{1}M_m^2\frac{F^2}{\varphi^2}-\frac{\rho_m}{\rho_\infty}=\left(1+\frac{\varkappa-1}{2}M_m^2-\frac{\rho_m}{\rho_\infty}\right)\sqrt{\frac{\rho_m}{\rho}}\,\theta,$$

or

$$r^2-\left(1+\frac{\varkappa-1}{2}M_m^2-\frac{\rho_m}{\rho_\infty}\right)\theta r-\frac{\rho_m}{\rho_\infty}+\frac{\varkappa-1}{2}M_m^2\frac{F^2}{\varphi^2}=0,$$

and hence

$$r=\left(1+\frac{\varkappa-1}{2}M_m^2-\frac{\rho_m}{\rho_\infty}\right)\frac{\theta}{2}\left[1\pm\sqrt{1+\frac{4\left(\frac{\rho_m}{\rho_\infty}-\frac{\varkappa-1}{2}M_m^2\frac{F^2}{\varphi^2}\right)}{\theta^2\left(1+\frac{\varkappa-1}{2}M_m^2-\frac{\rho_m}{\rho_\infty}\right)^2}}\right]$$

(where $\varkappa=\frac{c_p}{c_v}$).

Note that with given value of the ratio $\frac{\rho_m}{\rho_\infty}$ the increase of Mach's number M_m results qualitatively in the same final result (when calculating the velocity distributions, etc.) as a precooling of the gas in the jet.

In this way, also here a calculation of the function $T_{em}(x)$ is analogous) the calculation can be finished.

13.5. TRANSFORMATION OF THE EQUATIONS

As shown above, the mathematical procedure of calculating turbulent gas jets is based on the supposition that in self-similar flows the universal distributions of momentum flux density ρu^2, of surplus heat content $\rho u c_p \Delta T$ and mass $\rho u \Delta c$ are described by one and the same nondimensional expressions, functions of the argument $\varphi=\frac{y}{ax}$. We are

interested in the "price" of obtaining such a result for an earlier phase, that is, which assumptions are necessary for this. In other words, we are interested in the possibility of such a transformation of the system of equations of a turbulent boundary layer of compressible gas to new variables U, V, etc., such that the form of the equations and boundary conditions in the new variables remains unchanged compared to the initial equations and boundary conditions in the old var ables, u, v, etc., for the incompressible fluid.

It is obvious that, having carried out this transformation, we can go still farther and apply the apparatus of one of the semi empirical theories (e.g., the method of the asymptotic boundary layer used in this book) to the transformed system of equations. In this case, the final formulas in the new variables must also remain unchanged.

Following [55, 58] we first discuss the fundamental physical assumptions which we shall use. For simplicity we restrict ourselve to the case of a nonisothermal jet of low velocity. We shall consider in detail the particular features of the case $M \gtrsim 1$

In the case of an evolved turbulent motion these assumptions result in the following:

1. In the transformed equations all terms which refer to effects of molecular transfer (containing the coefficients of viscosity and thermal conductivity) will be considered as negligibly small compared with the analogous terms referring to molar turbulent exchange.

2. We shall adopt approximately that the gas density ρ is only a function of the temperature: $\rho \approx \frac{\text{const}}{T}$ (note that with $M \ll 1$ the limitation $\frac{\Delta p}{p} \ll 1$; is sufficient; in the case of $M \gtrsim 1$ we need the more rigid condition of $p \approx$ const).

3. Assuming that, as already indicated, the main part in turbulent

exchange is played by the pulsation of the "fused" quantities ρu^2, etc., we shall apply with the new variables U, V, J, etc., a separation of the averaged and pulsatory values of the quantities.

4. The motion will be considered as steady on the average, for both the mean values $\overline{U}$, etc., and the averaged pulsatory characteristics of the form of $\overline{U'V'}$ etc.

For the sake of generality, we shall demonstrate the transformation of equations to the new variables with the Navier-Stokes equations as an example, though an analogous transformation of the boundary layer equations is a little simpler. It should, however, be noted that the simplest way of arriving directly at the sought result consists in a transformation to the new variables and a subsequent averaging of Euler's equations. This would mean to omit *a priori* in the equations all those terms which represent the influence of molecular transfer (viscosity and thermal conductivity). In spite of the fact that in the given case, the final result remains the same, we shall proceed in a stricter way and discard the terms of molecular nature not at the beginning but at the end of the derivation, i.e., after the averaging.

The system of the equations of motion, continuity and energy (without volume forces and heat sources and with M << 1) we can write the expressions for the actual quantities in the form

$$\left.\begin{aligned}
&\rho\frac{\partial u_i}{\partial t}+\rho u_j\frac{\partial u_i}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_j}\left(\mu\frac{\partial u_i}{\partial x_j}\right), \quad i, j=1, 2, 3,\\
&\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_j}(\rho u_j)=0,\\
&\rho c_p\frac{\partial \Delta T}{\partial t}+\rho c_p u_j\frac{\partial \Delta T}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\lambda\frac{\partial \Delta T}{\partial x_j}\right)
\end{aligned}\right\} \qquad (13.1)$$

(summation, as always, over repeated subscripts).

When we join the equations of motion and energy (each one separately) with the continuity equation and pass over to the new variables $U_i=\sqrt{\rho}\,u_i, J=\sqrt{\rho}\,c_p\Delta T$ etc., we obtain

$$
\left.\begin{aligned}
\frac{\partial}{\partial t}(\sqrt{\rho}U_i)+\frac{\partial}{\partial x_j}(U_iU_j)&=-\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_j}\left(\mu\frac{\partial u_i}{\partial x_j}\right),\\
\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x_j}(\sqrt{\rho}\,U_j)&=0,\\
\frac{\partial}{\partial t}(\sqrt{\rho}J)+\frac{\partial}{\partial x_j}(U_jJ)&=\frac{\partial}{\partial x_j}\left(\lambda\frac{\partial\Delta T}{\partial x_j}\right).
\end{aligned}\right\}\qquad(13.2)
$$

Note for the following that from the energy equation written for T:

$$
\frac{dT}{dt}=\frac{1}{\rho c_p}\frac{\partial}{\partial x_j}\left(\lambda\frac{\partial T}{\partial x_j}\right),
$$

after averaging and neglecting the molecular effects in the averaged motion, it follows that $\frac{d\overline{T}}{dt}\approx 0$, and also

$$
\frac{d\overline{\sqrt{\rho}}}{dt}\approx 0\left(\text{since } \frac{d\sqrt{\rho}}{dt}=\frac{\sqrt{\rho}}{2T}\frac{dT}{dt}\right).
$$

Applying the operation of averaging with the equality $\frac{d\overline{\sqrt{\rho}}}{dt}\approx 0$, taken into account and also $\frac{\partial\overline{U}_j}{\partial x_j}=0$ (which results from the continuity equation) we finally arrive at the following system of equations:

$$
\left.\begin{aligned}
&\overline{U}_j\frac{\partial\overline{U}_i}{\partial x_j}=-\frac{\partial\overline{p}}{\partial x_i}-\frac{\partial}{\partial x_j}\overline{(U_i'U_j')},\qquad i=1,2,3,\\
&\frac{\partial\overline{U}_j}{\partial x_j}=0,\qquad \overline{U}_j\frac{\partial\overline{J}}{\partial x_j}=-\frac{\partial}{\partial x_j}\overline{(U_j'J')}.
\end{aligned}\right\}\qquad(13.3)
$$

At a first sight, the system of equations (13.3) in the new variables coincides with the analogous system for the incompressible fluid in the old variables u, v, etc. It is obvious that if we replace in this system the terms $-\overline{U_i'U_j'}$ and $-\overline{U_j'J'}$ by the semiempirical expressions for turbulent friction and heat flux (e.g., in the form of a linear relation between the turbulent stress tensor τ_{Tij} and the tensor of the "deformation rate" $\frac{\partial\overline{U}_i}{\partial x_j}$), we again obtain a corresponding generalization of Reynolds' equations for ρ = const to the case of a compressible gas.

If instead of this, we pass over from System (13.3) to the corresponding system of the turbulent boundary layer equations we may apply also to them an arbitrary semiempirical method (the mixing path method

in Prandtl's or Taylor's form, the method of the asymptotic layer and dimension considerations, etc.). In this case, an arbitrary concrete problem on self-similar jet flows solved for an incompressible fluid is generalized to the case of a gas by way of an algebraic conversion of the final solution (i.e., a replacement of u, v, ΔT, ... by U, V, J, ...).The results of such a transition when returning to the velocity, etc., has been discussed above in detail.

The method of similarity of ρu^2 for self-similar jets, in particular, the supposition of the decisive part of the momentum flux density transfer etc., is closely related with the method of calculation with the help of the equivalent problem of the theory of thermal conductivity to which we shall devote the next chapter. In this method, which permits the extension of the approximate solution of the problem of a turbulent jet of fluid or gas to a nonselfsimilar flow, taking into account the influence of the initial distribution and its subsequent continuous distortion, the results for the self-similar region are obtanned by means of a limiting transition with $\frac{s}{d} \to \infty$. The results of the calculation, their comparison with experimental data and the discussion of the less carefully investigated problem of the values of the experimental constants and the influence various factors exert on them will therefore be considered expediently after we have become familiar with this method.

REFERENCES

2, 3, 4, 7, 10, 11, 12, 25, 27, 28, 31, 41, 47, 48, 49, 50, 55, 58, 61, 64, 69, 73, 74, 78, 80, 82, 83, 96, 114, 122, 140, 145, 146, 175, 186, 193, 211, 212, 217, 218, 230, 236, 237, 240, 241, 242, 245, 249, 263, 269, 270, 276, 285, 308, 314.

Manuscript Page No.	[Footnotes]
292	Here and in the following we consider the case of absence of mass forces.
295	Unfortunately, this experiment is extremely limited (see [200, 247, 269, etc.]).
298	Here and also in other places (see note on page 234) the mean value $\overline{\rho u^2}$ was considered to be approximately equal to ρu^2, etc.

Chapter 14

THE METHOD OF THE EQUIVALENT PROBLEM OF THE THEORY OF THERMAL CONDUCTIVITY

14.1. ON THE LINEARIZATION OF THE FREE BOUNDARY LAYER EQUATIONS

Among the various mathematical semiempirical methods applicable to self-similar turbulent jets of incompressible fluid a somewhat isolated place is taken by the methods of calculation which are based on a replacement of the boundary layer equations by linear equations of the type of the heatconduction equation. Beginning with the choice of the empirical expressions for the velocity distribution in an axisymmetrical source jet of the form $\frac{u}{u_m} = \exp\left(-\frac{1}{2}\varphi^2\right)$, which are particular solutions to the heatconduction equation on the basis of a formal resemblance of the curves [153, 158, etc.], up to more or less successive integrations of an equation of the type of the heatconduction equation for the dynamic pressure [144, 282, 283, 322, etc.], these methods become more and more widespread at present.

The advantage connected with the transition from nonlinear, partial differential equations to linear ones is, of course, beyond any doubt. But the physical and mathematical fundamentals of this replacement and the laws for their application require a careful consideration.

The phenomenological (intuitive or "inductive" as it has already been called" theory of the type of the theory of thermal conductivity suggested by H. Reichardt [282], which is most completely developed, is discussed in relative detail in monographs which contain sections devoted to free turbulence [11, 200, 212, 214], though some orthodox

researchers of the boundary layer ignore it. As a rule, discussions of Reichardt's theory result in stating that, on the one hand, its application is successful but that it is physically (and also mathematically) unfounded on the other. Recall in this connection that for the derivation of the fundamental equation of this theory one introduced an arbitrary "law of thermal conductivity" for the momentum transfer, reading

$$\overline{uv} = -\Lambda(x)\frac{\partial u^2}{\partial y}.$$

For self-similar problems the coefficient of "thermal conductivity" $\Lambda(x)$ is chosen from dimension considerations: $\Lambda(x) = Cx$, C being an empirical constant. In H. Reichardt's first papers, the calculation was carried out for several cases of self-similar flows of incompressible fluid, turbulent jets and wake, which were solved previously by other semiempirical methods. With appropriate choice of the constant C the calculation was led to satisfactory agreement with experiment and with other solutions.

In P.V. Melent'yev's paper [144] a physically interesting attempt was made of considering the scattering of momentum flux density ρu^2 (in self-similar jets of incompressible fluid) as a probable process with the normal distribution function for random processes. The choice of the latter predetermines the reduction of the fundamental equation (of momentum "diffusion") to the form of a heatconduction-type equation. A full value derivation of this equation on the basis of the modern statistical theory of turbulence which would be of great interest, is not available.

In some papers one arrives at the heatconduction-type equation for the dynamic pressure by way of an arbitrary unjustified neglection of terms in the boundary layer equation (terms of the same order as those

left over) and a subsequent artificial selection of an expression for the turbulent viscosity ν_T. These papers will not be discussed here.

As regards the application of the "heat-conduction" equation in calculations, in a series of papers [322, etc.], the authors consider problems on parallel flows and the like which are not comprised within the framework of self-similar jets. Here too the absence of a rational theory results in nonjustified simplifications, is contradictory, essentially with the basic laws of fluid mechanics. For example, in individual papers in the equation

$$\frac{\partial u^2}{\partial x} = Cx\frac{\partial^2 u^2}{\partial y^2}$$

and in the boundary conditions a transition to parallel or antiparallel motions is carried out by way of replacing u^2 by $u^2 \pm u_0^2$, though the actual motion does not agree with this: from the physical point of view the efflux of a jet in a quiescent or moving fluid is not equivalent, from the mathematical one, we are concerned with problems with different boundary conditions. It does not correspond to experimental results and dimension considerations either when the expression for the coefficient $\Lambda(x) = Cx$, which applies to source jets, is transferred to a flow in an obviously nonselfsimilar region (e.g., a jet of finite dimensions). In this case, from the parameters describing the motion the characteristic dimensions are discarded.

In spite of all these remarks indicating the absence of a firm base for both the formulation and the application of a linear homogeneous equation of the type of the heatconduction equation instead of the nonlinear equations of the turbulent boundary layer, such a replacement results in most cases in a qualitative and, with some degree of accuracy, also quantitative and satisfactory agreement with experiment. Especially for self-similar jets of incompressible fluid and other

semiempirical schemes the agreement with experiment is achieved by choosing a single empirical constant. This fact, naturally, drew attention to discussions of the method, attempts of giving it a basis and to various applications.

In a series of papers [114, 149, 150, 193, 199] many examples of such applications of the heatconduction-type equation are given. In these papers, in order to simplify calculation, successful use was made of computers, hydrodynamic and hydrostatic integrators, static electro-integrators [34, 35, 142, etc.] and recently (for the axisymmetric problem) numerical solutions were obtained with the help of the so-called *P*-function tabulated by J. Masters [264].

Besides this the theory of a method was developed [63, 65] which was called the method of the equivalent problem of the theory of thermal conductivity. The difference between this method and the initial scheme (H. Reichardt et al.) consists in the fact that it is reduced to an essentially ordinary (but also continuous in its form) approximate mathematical transformation of variables. We are in this case concerned with a linearization of the problem which is achieved by a peculiar replacement of the variables in the turbulent boundary layer equations. In this replacement, — and this is the peculiarity —, the connection between the new and the old independent variables is chosen according to experimental data. A transformation of this type and the pertinent transition from the physical plane of variables to the effective one (in which the equations are linear) can be applied to jets of incompressible fluid or compressible gas, to complex jet flows, parallel jets, jets of finite dimensions, etc. In the particular case of self-similar jets, a single empirical constant is taken from experiment. In more comples nonselfsimilar jets, the volume of information taken from experiment is higher. A rational application of the method is in concrete

cases determined by a peculiar optimum of this information and the subsequent simplification of the calculation. The application of the method proved very fruitful, for example, in the calculation of the problem of diffusion combustion of gas (see Chapter 17) which is important in practice, and others.

Thus, other than the first imperfect attempts of reducing the problems of the theory of the free turbulent boundary layer to the solution of a heatconduction-type equation, which required the introduction of additional (as a rule, physically voluntary or mathematically unjustified) hypotheses and assumptions, the method considered is as much founded as other semiempirical methods of calculation. Its application permits an essential widening of the field of mathematical problems of the theory of turbulent jets of fluid or gas and even – when the inverse problem is solved [65] – permits in principle the determination of approximate laws governing the turbulent friction and the heat exchange in nonselfsimilar jets, i.e., there where the usual formula (of the "mixing length") is known to be inapplicable.

14.2. REPLACEMENT OF VARIABLES FOR SELF-SIMILAR FLOWS

Let us write the initial fundamental equations of the laminar or turbulent boundary layer of compressible gas in the following form:

$$\left.\begin{aligned} &\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{1}{y^k} \frac{\partial}{\partial y} (y^k \tau),\\ &\frac{\partial}{\partial x} (\rho y^k u) + \frac{\partial}{\partial y} (\rho y^k v) = 0,\\ &\rho c_p u \frac{\partial T}{\partial x} + \rho c_p v \frac{\partial T}{\partial y} = \frac{1}{y^k} \frac{\partial}{\partial y} (y^k q). \end{aligned}\right\} \qquad (14.1)$$

Let us assume that an introduction of the new independent variables

$$\xi = \xi(x, y), \quad \eta = \eta(x, y)$$

for the dynamic problem and

$$\xi_T = \xi_T(x, y), \quad \eta_T = \eta_T(x, y)$$

for the thermal problem makes it possible to reduce the system of equations (14.1) to equations of the form

$$\frac{\partial}{\partial \xi}(\rho u^2) = \frac{\partial^2}{\partial \eta^2}(\rho u^2) \tag{14.2}$$

and

$$\frac{\partial}{\partial \xi_T}(\rho u c_p \Delta T) = \frac{\partial^2}{\partial \eta_T^2}(\rho u c_p \Delta T) \tag{14.3}$$

with boundary conditions transformed correspondingly. In such a general form, the problem of an analytical determination of the functional relationship between old and new independent variables is obviously hopeless. In the case of turbulent motion, it is yet worse since the expressions for the turbulent frictional stress and heat flux, i.e., for the functions τ_T and q_T in the initial equations are unknown in the general case.

We shall therefore begin with simpler cases, where the problem is easy to solve [65]. As a first example, we consider the problem of the expansion of laminar source jets (plane or axisymmetric) of an incompressible fluid. Solutions of these problems were already discussed in detail and are compiled in Table 7.1. Therefore, omitting the transformations, we give side by side the two solutions for each of the problems: the first one is the same as before, on the lefthand side; the second has been transformed to an equation of the heatconduction-type, on the righthand side.

I. Solution obtained by the method of the asymptotic layer.	II. Solution obtained by the method of the equivalent problem.
Plane source jet	
Initial equations	
$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$	$\frac{\partial u^2}{\partial \xi} = \frac{\partial^2 u^2}{\partial \eta^2}$
Boundary conditions	
$\frac{\partial u}{\partial y} = v = 0$ with $y = 0$, $u = 0$ with $y = \pm\infty$	$\frac{\partial u^2}{\partial \eta} = 0$ with $\eta = 0$, $u^2 = 0$ with $\eta = \pm\infty$

Integral invariants

$J_x = \int\limits_{-\infty}^{+\infty} u^2 dy = \text{const}$	$J_\xi = \int\limits_{-\infty}^{+\infty} u_1^2 d\eta_1 = \text{const}$

Self-similarity transformations

$\frac{u}{u_m} = F'(\varphi),\ u_m = Ax^{\alpha},$	$\frac{u_1^2}{u_{1m}^2} = \Phi(\psi),\ u_{1m}^2 = A_1\xi^{2\alpha_1},$
$\varphi = Byx^{\beta},$	$\psi = B_1\eta_1\xi^{\beta_1},$
$\alpha = -\frac{1}{3},\ \beta = -\frac{2}{3},$	$\alpha_1 = -\frac{1}{4},\ \beta_1 = -\frac{1}{2},$
$A = \frac{1}{2}\sqrt[3]{\frac{3J_x^2}{4\nu}},\ B = \frac{1}{2}\sqrt[3]{\frac{J_x}{6\nu^2}}$	$A_1 = \sqrt{\frac{J_\xi}{2\sqrt{\pi}}},\ B_1 = \frac{1}{2}$

Self-similar equations

$F''' + 2(FF')' = 0.$	$\Phi'' + 2(\psi\Phi)' = 0$

Boundary conditions

$F = 0,\ F' = 1$ with $\varphi = 0,$	$\Phi = 1$ with $\psi = 0,$
$F' = 0$ with $\varphi = \pm\infty$	$\Phi = 0$ with $\psi = \pm\infty$

Solutions of the equations

$F' = \frac{1}{\mathrm{ch}^2\varphi}$	$\Phi = \exp(-\psi^2)$

Comparing the two solutions, we require that the expressions for the relative velocity u/u_m and $\frac{u_1}{u_{1m}}$ coincide, i.e., we assume that the expression of $F'(\varphi)$ is equal to that of $\sqrt{\Phi(\psi)}$. From the equation

$$\frac{1}{\mathrm{ch}^2\varphi} = \exp\left(-\frac{\psi^2}{2}\right)$$

it follows that

$$\psi = \sqrt{2\ln \mathrm{ch}^2\varphi}.$$

From the condition $u_m(x) = u_{1m}(\xi)$, i.e., $Ax^{\alpha} = A_1\xi^{\alpha_1}$ and the last equation, we obtain the final transformation formula for the variables $\xi = \xi(x)$:

$$\xi = \left(\frac{A}{A_1}\right)^{\frac{1}{\alpha_1}} x^{\frac{\alpha}{\alpha_1}} = \frac{16}{9\pi}\frac{\nu J_\xi^2}{J_x^2}\left(\frac{4}{3}\nu J_x\right)^{1/3} x^{4/3}.$$

Thus, the new variables which reduce the initial system of equations of the laminar boundary layer of incompressible fluid to a parabolic equation in canonic representation, are, for the problem consider-

ed, given by the above formulas. This is easy to verify by way of direct substitution.

Note that after we have solved the heatconduction-type equation in the ξ, η plane and have returned to the physical x,y-plane, we can also determine the second velocity component $v = -\int \frac{\partial u}{\partial x}\, dy$. with the help of the continuity equation.

It attracts our attention that the transformation sought for the given problem has the form $\xi = \xi(x)$, $\eta = \eta(x, y)$, which is a general property of transformations for self-similar equations.

Let us also give a brief comparison of the two solutions for a laminar axisymmetric source jet of incompressible fluid.

Initial equations, boundary conditions and integral invariants

$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\nu}{y}\frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial y}\right),$	$\frac{\partial u_1^2}{\partial \xi} = \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta\frac{\partial u_1^2}{\partial \eta}\right),$
$\frac{\partial}{\partial x}(yu) + \frac{\partial}{\partial y}(yv) = 0,$	
$v = 0,\ \frac{\partial u}{\partial y} = 0$ with $y = 0,$	$\frac{\partial u_1^2}{\partial y} = 0$ with $\eta = 0,$
$u = 0$ with $y = \infty,$	$u_1^2 = 0$ with $\eta = \infty,$
$\int_0^\infty u^2 y\, dy = \text{const} = J_x$	$\int_0^\infty u_1^2 \eta\, d\eta = \text{const} = J_\xi$

Self-similar equations and their solutions

$\left(F'' - \frac{F'}{\varphi}\right)' + \left(\frac{FF'}{\varphi}\right)' = 0,$	$\frac{(\psi\Phi')'}{\psi} + \frac{1}{2}\Phi\psi' + \Phi = 0,$
$\frac{F'}{\varphi} = 1,\ \frac{F}{\varphi} = 0$ with $\varphi = 0,$	$\Phi' = 1$ with $\psi = 0,$
$\frac{F'}{\varphi} = 0$ with $\varphi = \infty,$	$\Phi = 0$ with $\psi = \infty,$
$\frac{u}{u_m} = \frac{F'(\varphi)}{\varphi} = \frac{1}{\left(1+\frac{1}{8}\varphi^2\right)^2},$	$\frac{u_1^2}{u_{1m}^2} = \Phi(\psi) = \exp\left(-\frac{1}{4}\psi^2\right),$
$u_m = Ax^a = \frac{3J_x}{8\pi\nu}\frac{1}{x}$	$u_{1m}^2 = A_1\xi^{2a_1} = \frac{1}{2}J_\xi\frac{1}{\xi}$

From this we see that the conditions of substitution of the variables for the laminar axisymmetric jet, which reduce the self-similar equations and their solutions to identity and which are obtained in the

usual way (lefthand corner) in the physical space of x, y and through the equivalent problem of the theory of thermal conductivity (right-hand column) in the "linear" space ξ, η, have the form

$$\xi = \frac{J_\xi}{2}\left(\frac{8\pi\nu}{3J_x}\right)^2 x^3, \quad \psi = 4\sqrt{\ln\left(1+\frac{1}{8}\varphi^2\right)}.$$

In this case too, $\xi = \xi(x)$, $\eta = \eta(x, y)$.

In this way, for two cases of a self-similar laminar jet flow, we can give (or rather find from a comparison of the solutions as shown above) formulas for the substitution of variables which transform the equations of the problem to a unidimensional (linear and homogeneous) equation of the heatconduction-type, in canonic representation. Precisely the same can be done with other problems on self-similar laminar jets. It is essential that the form of the transformations is connected with the boundary conditions of the problem. For example, with plane or round jets the formulas of substitution of variables obtained for the source-jet efflux in a nonmoving medium are inapplicable in the case of a jet which expands in a parallel flow (this would also be obvious from the fact that the problem of parallel jets cannot be reduced to a self-similar one).

Let us now turn to the application of the method of the equivalent problem of the theory of thermal conductivity to self-similar turbulent jet flows. Since in the theory of the asymptotic layer the equations of the self-similar jet flows and their solutions written in nondimensional variables agree for laminar and turbulent motions, the above transformation formulas of the self-similar variables $\psi = \psi(\varphi)$ are the same for turbulent self-similar jets of incompressible fluids:

For a plane source jet $\psi = \sqrt{2 \ln \operatorname{ch}^2 \varphi}$,

for an axisymmetric source jet $\psi = 4\sqrt{\ln\left(1+\frac{1}{8}\varphi^2\right)}$.

The values of the self-similarity constants α_1 and β_1 also re-

main unchanged for the heatconduction-type equations; in particular, in all cases $\beta_1 = -\frac{1}{2}$ a fact which is very important (this results immediately from the form of the canonic equation of dimensions which are not characteristic under the conditions of the self-similar problem such that the nondimensional variable $\varphi \sim \frac{\eta}{\sqrt{\xi}}$).

The values of the constants of self-similarity (and also the constants A and B) are, however, different for the left-hand sides which refer to the turbulent self-similar jets. Since in all cases, the constant $\beta = -1$, an equating of the quantities u_m and u_{1m} yields $\xi \sim x^2$ or $\xi = cx^2$ as the general law of transformation of longitudinal coordinates for self-similar turbulent jets.

As regards the second formula of substitution of variables $\eta = \eta(x, y)$, it can be found in each case from the formulas linking φ and ψ and also ξ and x. We can thus also here obtain a final solution of the problem of substitution of variables resulting in a heatconduction-type equation. The solution of the latter, just as one of the initial equations, contains a single empirical constant which enters, for example, the formula $\xi = cx^2$.

Thought with this the mathematical part of the problem is exhausted, it is expedient to compare the two formulas of solution of self-similar equations (for example, $F(\varphi) = \frac{1}{\mathrm{ch}^2\varphi}$ and $\sqrt{\Phi(\varphi)} = \exp\left(-\frac{\varphi^2}{2}\right)$ for the plane source jet and analogous formulas for other problems) and thus to find the approximate transformation which is more suitable for practical purposes.

In Fig. 14.1 we have compared the two solutions for plane and axisymmetric source jets in the coordinates $\frac{u}{u_m} = f\left(\frac{\varphi}{\varphi_{1/2}}\right)$ or $\frac{u}{u_m} = f_1\left(\frac{\psi}{\psi_{1/2}}\right)$ As we see from the figures, in both cases (and also with other self-similar problems) the curves obtained immediately from the boundary layer equations are very similar, even after the transition to the

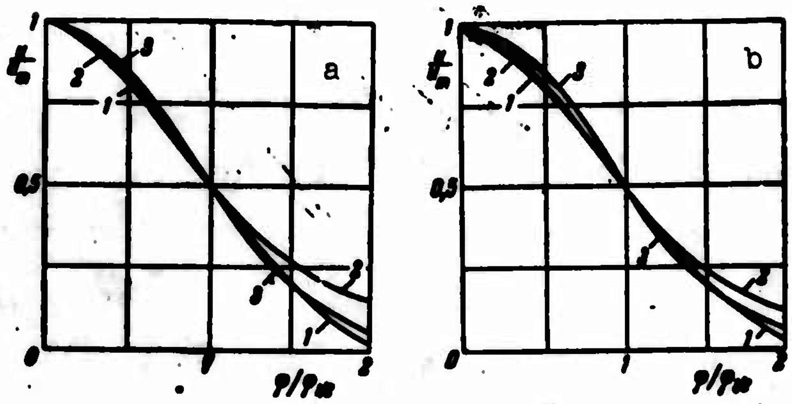

Fig. 14.1 Universal velocity profiles of plane (a) and round (b) source jets. 1) By finite-layer method; 2) by asymptotic-layer method; 3) by equivalent-problem method.

equivalent problem of the theory of thermal conductivity. In practice, as this can be seen from a comparison with experiment, the differences between the curves are within the limits of scatter of the experimental points. The curves which correspond to the solution of the equivalent problem agree even better with the experiment.

Since the comparison with experiment is achieved by means of the choice of an empirical constant, the actual agreement of the curves in Fig. 14.1 prompts the approximative substitution of the variables of the form $\Phi \sim \varphi$ or, taking $\sqrt{\xi} \sim x$, into account, of $\eta \approx y$. The empirical constant enters here the coefficient of proportionality in the equality $\xi = cx^2$.

This approximate result, namely the formula of substitution of variables for self-similar equations of turbulent jets of incompressible fluid, might have been predicted earlier, on the basis of dimension considerations. In fact, since the equation $\beta = -1$ corresponds to linear effective boundaries of the jet $(b \sim y_{1/2} \sim x)$, the equation $\beta_1 = -1/2$ (i.e., $b_1 \sim \eta_{1/2} \sim \sqrt{\xi}$) results in a proportionality of the effective bound-

aries of the jet: $\psi_{1/2} \sim \varphi_{1/2}$. The latter equation (taking into account that $\sqrt{\xi} \sim x$) can be satisfied if we put $y_{1/2} = \eta_{1/2}$ or generally $y \approx \eta$.

Thus, with an accuracy corresponding to the spread of the experimental points, the formulas of the approximate transformation to the equivalent problem of the theory of thermal conductivity for self-similar turbulent jets of incompressible fluid have the simple form of

$$\xi = cx^2, \quad \eta = y, \quad \psi \sim \varphi.$$

14.3. THE GENERAL CASE OF TURBULENT JETS OF FLUID AND GAS

Let us now generalize the results obtained in the last section to self-similar jets of compressible gas firstly and to nonself-similar turbulent jets of fluid or gas secondly. The latter will correspond to the most general case of application of the method of the equivalent problem of the theory of thermal conductivity. At the same time, we show types of problems for which the application of this method is effective and establish the forms of expressions for the turbulent stress of friction and the heat flux in the nonselfsimilar region in agreement with the chosen form of transformation.

For self-similar jets of compressible gas, it is natural to use the same form of substitution of variables as in the case of the incompressible fluid as this could be shown by the above considerations. The difference consists of the fact that from the equations

$$\left.\begin{aligned} \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} &= \frac{1}{y^k} \frac{\partial}{\partial y} (y^k \tau_T), \\ \frac{\partial}{\partial x} (\rho y^k u) + \frac{\partial}{\partial y} (\rho y^k v) &= 0 \end{aligned}\right\} \tag{14.4}$$

one must pass over to the equivalent equation* for ρu^2 of the form

$$\frac{\partial}{\partial \xi} (\rho u^2) = \frac{1}{\eta^k} \frac{\partial}{\partial \eta} \left(\eta^k \frac{\partial (\rho u^2)}{\partial \eta} \right), \tag{14.5}$$

where $\xi = cx^2, \eta \approx y$. The constant c is obtained from experiments.

Analogously, from the equation

$$\rho u c_p \frac{\partial \Delta T}{\partial x} + \rho v c_p \frac{\partial \Delta T}{\partial y} = \frac{1}{y^k} \frac{\partial}{\partial y}(y^k q_T) \qquad (14.6)$$

with the substitution of the variables $\xi_T = c_T x^2, \eta_T \approx y$ one must pass over to an equation of the heatconduction-type for the flux density of surplus heat content; in the case of a high-velocity jet both equations can be formulated for the total enthalpy $(c_p T_0 = c_p T + \frac{u^2}{2})$:

$$\frac{\partial}{\partial \xi_T}(\rho u c_p \Delta T_0) = \frac{1}{\eta_T^k} \frac{\partial}{\partial \eta_T}\left[\eta_T^k \frac{\partial}{\partial \eta_T}(\rho c_p u \, \Delta T_0)\right]. \qquad (14.7)$$

Having obtained a solution to heatconduction-type equations, it is easy to pass over to formulas for the velocity and the surplus temperature with the help of an algebraic reduction, analogous to that given in the section dealing with the calculation according to the method of similarity of ρu^2, etc. In this respect, the method of the equivalent problem applied to self-similar turbulent jets of gas does not give any new aspects.

An effective application of this method is connected with an approximate solution of the nonself-similar problems of the theory of turbulent jets of incompressible fluids and gases. Precisely this, namely the possibility of calculating a jet of finite dimensions with arbitrary form of the initial distribution of ρu^2 and $\rho u c_p \Delta T$ (velocity and surplus temperature with ρ = const) of the jet expanding in a parallel flow (without the supposition of small perturbations) etc., is the main advantage of this method.

In the general case, we are concerned with the determination of the formulas of substitution of variables for the transition from Eqs. (14.4) and (15.6) to Eqs. (14.5) and (14.7) with the essential difference that the expressions for turbulent friction and heat flux are unknown for nonself-similar motions. In fact, none of the semiempirical formulas used previously for the calculation of selfsimilar

jets can here be applied.

Since the initial boundary layer equations belong to the class of nonlinear parabolic equations, it is natural to assume that the general formulas for substitution of variables of the form

$$\xi = \xi(x, y), \quad \eta = \eta(x, y),$$

and the more particular formulas of substitution of variables of the form

$$\xi = \xi(x), \quad \eta = \eta(x, y)$$

prove to be applicable to transformation of Eqs. (14.4) and (14.6) to the canonic form of (14.5) and (14.7). The concrete form of the formulas of substitution of the variables may in each of the cases given be chosen by way of comparing the solutions of the linear heat-conduction-type equations with the experimental distributions of ρu^2 and $\rho u c_p \Delta T$ in the coordinates x, y.

In this form (general or particular) the transformation, which requires the establishment of a mutually unambiguous correspondence of the variables ξ, η with the true coordinates x, y, might only be valuable in practice after a detailed experiment has been carried out, as the result of a gathering and generalization of the vast experimental and theoretical material.

At present, it is much more effective to choose the narrower path of using still less general approximation formulas for the substitution of the variables. Let us assume that for a certain class of jet flows of fluid or gas (the experiment shows that the flows mentioned above belong to it, i.e., the jet of finite dimensions, the jet in the parallel flow and the like) the conditions of substitution of variables, which result in equations of the type of the heatconduction equation, read

$$\xi = \xi(x), \quad \eta \approx y,$$

and also

$$\xi_T = \xi_T(x), \quad \eta_T \approx \eta \approx y$$

for Eqs. (14.5) and (14.7), respectively. In this case, it is possible, from a comparison of the results of a particular experimental investigation with the analytical solution, for example, by way of comparing the experimental dependence $\frac{(\rho u^2)_m}{(\rho u^2)_0} = f\left(\frac{x}{d}\right)$ and the theoretical one $\frac{(\rho u^2)_m}{(\rho u^2)_0} = F\left(\frac{\sqrt{\xi}}{d}\right)$ to establish a relation linking ξ and x. If the supposition of $\eta \approx y$ corresponds to the given problem, in the case of a combination of the coordinates x and ξ the experimental distributions of ρu^2 in the jet cross sections must coincide more or less accurately with the calculated curves. In this case the above form of transformations is effective since the empirical admixture is relatively small. The more data we possess on the function $\xi = \xi(x)$ for the various problems, the more successful are the possibilities, though only at first, of precalculating approximately the laws governing the expansion of jets.

Note that an analytical (or, with the help of integrators, numerical) solution of heatconduction-type equations is developed with the same boundary conditions with which Eq. (14.4) should have been integrated. The results from the fact that the coordinates η and y and, consequently, also the boundary conditions with $y = 0$ and $y = \infty$ and, correspondingly, with $\eta = 0$ and $\eta = \infty$, coincide.

The same holds true for the "initial" conditions with $x = 0$ (or $x \to \infty$) since $\xi = 0$ and $\xi \to \infty$ corresponds to the value $x = 0$ and $x \to \infty$. In this case the solution of the linear equation, which permits a superposition of the particular solutions, has great advantages compared with the initial nonlinear problem.

If $\xi = \xi(x)$ is known from experiment it is easy to reduce the solution obtained in the variables ξ, y to the variables x and y. This

function has an especially simple form in the region of motion far away from the source. In this case, the flow is self-similar and, obviously, we must pass over from the equation of the form of $\xi = \xi(x)$ to one of the form of $\xi = cx^2$ which contains a single empirical constant.

At the end of this problem, we again have to carry out an analogous procedure to find the relation linking the coordinates ξ_T and x for the thermal problem, using the experimental function $\frac{(\rho u c_p \Delta T)_m}{(\rho u c_p \Delta T)_0} = F\left(\frac{x}{d}\right)$ After the combination of ξ_T and x the transverse distributions of $\rho u c_p \Delta T$ coincide ($\eta \approx y$, if this transformation corresponds with sufficient accuracy to the flow given). In order to solve the thermal problem, however, other ways may also prove to be convenient, for example, in the theory of jets of incompressible fluid, the way of direct integration of the linear equation of heat transfer, with the velocity distribution obtained from a solution of the dynamic problem and knowing the additional empirical constant Pr_T. But also for a solution by the method of the equivalent problem it is possible to achieve an essential simplification if, instead of measuring in the experiments the value of $(\rho u c_p \Delta T)_m$ on the jet axis the form of the relation linking the "longitudinal coordinates" ξ_T and ξ is postulated. For example, in the case of self-similar jets of fluid and gas the ratio ξ/ξ_T which might be considered as an analogue to the turbulent Prandtl number in the method of the equivalent problem, is approximately constant and even numerically close to the value of Pr_T $\left(\frac{\xi}{\xi_T} \approx 0{,}7 \div 0{,}8\right)$.

Examples of solutions and a comparison with experimental data will follow in the next chapter.

14.4. THE TURBULENT STRESS OF FRICTION AND THE HEAT FLUX

For this class of jet flows, for which only the longitudinal coordinate x must be transformed while the transverse coordinate is invar-

iant, we can, following [65], given the corresponding expressions for the turbulent stress of friction and the heat flux.

In fact, let us set equal Eqs. (14.4) represented in the form

$$\frac{\partial}{\partial x}(\rho u^2)+\frac{1}{y^k}\frac{\partial}{\partial y}(y^k\rho uv)=\frac{1}{y^k}\frac{\partial}{\partial y_1}[y^k\tau_T(x, y)], \tag{14.8}$$

where $\tau_T(x,y)$ is a generally unknown function of the coordinates, and the canonic equation

$$\frac{\partial}{\partial \xi}(\rho u^2)=\frac{\partial}{\partial x}(\rho u^2)\frac{dx}{d\xi}=\frac{1}{y^k}\frac{\partial}{\partial y}\left[y^k\frac{\partial}{\partial y}(\rho u^2)\right]. \tag{14.9}$$

From the equality of the derivatives $\frac{\partial}{\partial x}(\rho u^2)$ in the two equations we find

$$\tau_T=\frac{\partial}{\partial y}(\rho u^2)\frac{d\xi}{dx}+\rho uv. \tag{14.10}$$

This two-term expression for the turbulent stress of friction is interesting from the physical point of view as besides the "gradient" term it also takes into account the convective one*.

Since

$$y^k\rho v=-\frac{d\xi}{dx}\int\frac{\partial}{\partial \xi}(\rho y^k u)\,dy,$$

results from the equation of continuity, we can give Eq. (14.10) the form of a product of $\frac{d\xi}{dx}=f(x)$ by a function which depends on ξ and y. When processing experimental data, the latter may be obtained from an analytical solution of the linear heatconduction-type equation where the boundary conditions have been taken into account, and the derivative $d\xi/dx$ is obtained from experiment. Analogously, by way of comparing the equations of heat transfer in the form

$$\frac{\partial}{\partial x}(\rho uc_p\Delta T)+\frac{1}{y^k}\frac{\partial}{\partial y}(y^k\rho uc_p\Delta T)=\frac{1}{y^k}\frac{\partial}{\partial y}[y^k q_T(x, y)] \tag{14.11}$$

and

$$\frac{\partial}{\partial \xi_T}(\rho uc_p\Delta T)=\frac{\partial}{\partial x}(\rho uc_p\Delta T)\frac{dx}{d\xi_T}=\frac{1}{y^k}\frac{\partial}{\partial y}\left[y^k\frac{\partial}{\partial y}(\rho uc_p\Delta T)\right] \tag{14.12}$$

we obtain a two-term expression for the turbulent heat flux

$$q_T = \frac{\partial}{\partial y}(\rho u c_p \Delta T)\frac{d\xi_T}{dx} + \rho v c_p \Delta T. \tag{14.13}$$

Just as τ_T, the quantity q_T consists of two terms, the terms of the "gradient" and the convective transfer.

The same expressions can be obtained for turbulent diffusion and also for the flux of total enthalpy (if $c_p \Delta T$ in Eq. (14.13) is replaced by $c_p \Delta T_0$).

The expressions obtained refer to submerged gas jets and, in the particular case of ρ = const, to jets of incompressible fluid.

In order to calculate a jet moving in a parallel flow (with the velocity u_∞) the heatconduction-type equation for the excessive momentum flux density $\rho u(u - u_\infty)$, is used as it satisfies the condition of conservation. The expression for turbulent friction is changed correspondingly: instead of ρu^2 it contains the term $\rho u(u - u_\infty)$. As shown in the evaluation of the experiments (s-e below) it is, however, possible to use as an approximation method suitable in practice the formulas of the dynamic problem given in the text in order to calculate parallel jets; for the application of these formulas it is sufficient to measure the distribution of the single quantity ρu^2 (when we pass over to the quantity $\rho u(u - u_\infty)$ we must also measure the temperature).

Let us also use Formulas (14.10) and (14.13) in order to formulate expressions for the turbulent viscosity μ_T and the thermal conductivity λ_T assuming that these characteristics are both finite in the whole field of flow. Dividing the expressions for τ_T and q_T by $\frac{\partial u}{\partial y}$ and $\frac{\partial \Delta T}{\partial y}$, respectively, we obtain for the coefficient of turbulent viscosity

$$\mu_T = \frac{\partial}{\partial u}(\rho u^2)\frac{d\xi}{dx} + \rho v \frac{\partial y}{\partial \ln u} \tag{14.14}$$

and for the coefficient of turbulent thermal conductivity

$$\lambda_T = \frac{\partial}{\partial T}(\rho u c_p \Delta T)\frac{d\xi_T}{dx} + \rho v c_p \frac{\partial y}{\partial \ln \Delta T}. \qquad (14.15)$$

For an incompressible fluid, these equations can be simplified a little and take the following form:

$$\nu_T = u\left(2\frac{d\xi}{dx} + v\frac{\partial y}{\partial u}\right) \qquad (14.16)$$

for the coefficient of turbulent kinematic viscosity and

$$a_T = u\left(1 + \frac{\partial \ln u}{\partial \ln T}\right)\frac{d\xi_T}{dx} + v\frac{\partial y}{\partial \ln T} \qquad (14.17)$$

for the coefficient of turbulent thermal diffusivity.

The ratio of these expressions defines the so-called turbulent Prandtl number.

In order to estimate to what extent these expressions for the tranfer coefficients and the quantity τ_T correspond to the known laws of turbulent transfer of momentum and heat in free jets of incompressible gas, the ratios $\frac{\tau_T}{\tau_{T\,max}}$ and $\frac{\nu_T}{\nu_{T\,max}}$ are compared in Fig. 14.2 as functions of the coordinate $\varphi = \frac{y}{ax}$ for the cross sections of an axisymmetric turbulent source jet of incompressible fluid.

As we see from the figure, the curves calculated according to Eqs. (14.10) and (14.16) are relatively close to the curves which refer to the method of the layer of finite thickness and Prandtl's formula $\tau_T = \rho c^2 x^2 \left(\frac{\partial u}{\partial y}\right)^2$, on the one hand and the calculation according to the method of the asymptotic layer and Prandtl's second formula $\tau_T = \rho b u_m \left(\frac{\partial u}{\partial y}\right)$ on the other. This, however, was to be expected since the universal velocity distribution calculated by the method of the equivalent problem of the theory of thermal conductivity resembles the universal distributions corresponding to this scheme (see Fig. 14.1).

Let us give yet another example of calculating the turbulent stress of friction by means of the two-term formula (14.10).

Figure 14.3 shows the distributions of the relative quantity of turbulent frictional stress calculated by I.B. Palatnik according to experimental data* by B.P. Ustimenko [199] for several cross sections

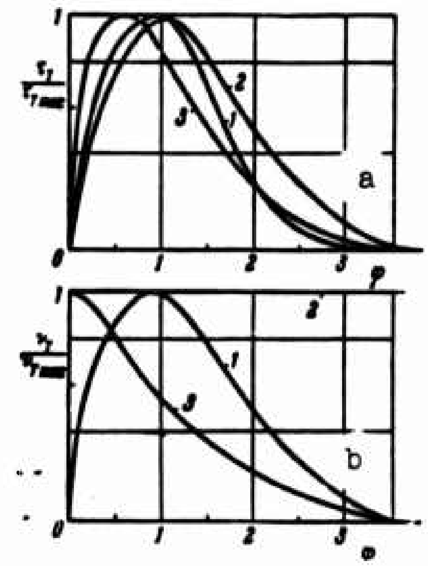

Fig. 14.2. Distribution of turbulent tangential stress of friction (a) and effective kinematic viscosity (b) in cross section of source jet. 1 – After the old Prandtl formula; 2 – after the new Prandtl formula; 3 – according to the method of the equivalent problem.

of a plane turbulent air jet discharged from a nozzle of finite dimensions. The curves in this figure (in contrast to Fig. 14.2 we consider here a nonself-similar flow) show clearly the displacement of the maximum of τ_T toward the jet axis with increasing distortion of the initial velocity distribution (which is uniform at the outlet of the jet). An analogous example for an axisymmetric gas jet will be given later (in Section 15.5) when comparing the results calculated by the method of the equivalent problem with experimental data.

The expressions given for τ_T and q_T is of interest, particularly with respect to nonself-similar jets. As regards the formulas for the coefficients ν_T and a_T, they may be applied as additional ones to the expressions for friction and heat flux, of course with the exception of those points in the flow where the derivates $\frac{\partial u}{\partial y}$ or $\frac{\partial T}{\partial y}$ vanish while τ_T or q_T are nonzero. Without pausing to consider this in detail, we only mention that from Eqs. (14.16) and (14.17) in a case where, for

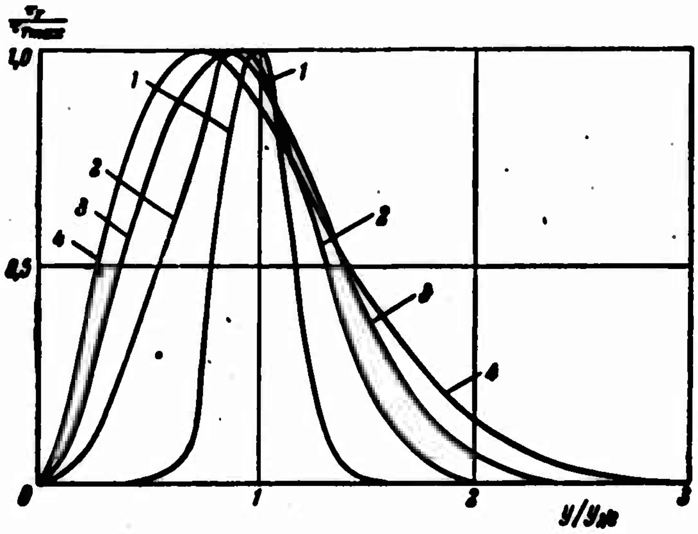

Fig. 14.3. Distribution of the turbulent stress of friction in the cross sections of a plane jet of finite dimensions.

TABLE A

№	1	2	3	4
x/b	0,5	2,5	7	20÷25

simplicity, the second terms can be neglected, an approximate expression results for the conventional effective turbulent "Prandtl number"

$$\mathrm{Pr}_T \approx \frac{2}{1+\frac{\partial \ln u}{\partial \ln T}} \frac{d\xi}{d\xi_T}. \tag{14.18}$$

In the self-similar section of a plane or axisymmetric source jet $\xi = cx^2$, $\xi_T = c_T x^2$, $\frac{c}{c_T} \approx 0{,}8$; moreover, $u \sim (\Delta T)^n$, where $n \simeq 1.5$. Thus, Eq. (14.18) yields $\mathrm{Pr}_T < 1$ and, roughly approximate $\mathrm{Pr}_T \approx 0{,}7$ which agrees with experiment.

This estimate also shows that the method of the equivalent problem of the theory of thermal conductivity applied to the well-known self-similar turbulent jet flows of incompressible fluid yields fully satisfactory results.

As shown later, an approximate agreement between calculation and experiment is also maintained for gas jets, even nonself-similar ones.

Finally, we shall discuss the qualitative conditions of applicability of the method of the equivalent problem of the theory of thermal conductivity and, yet more briefly, the substitution formulas for the variables $\xi = \xi(x)$, $\eta \approx y$. In a few words, these conditions comprise two points. Firstly, we are concerned with an evolved turbulent exchange, i.e., flow conditions in which the molecular effects of transfer are negligibly small compared with the molar ones. Secondly, and this point predetermines the possibility of maintaining the transverse coordinate $y \approx \eta$ when passing over from the physical space to the effective one in which the equation of turbulent transfer is linear, the condition of applicability of the methods of the boundary layer theory. In particular, we are concerned with the practical isobaricity of the flow and the smallness of the transverse velocity component relative to the longitudinal.

An extension of the approximate limits of applicability of the method of the equivalent problem mentioned here to other types of turbulent jet flows (and similar ones) of fluid and gas requires great care. With this question, it is the experiment which gives the decisive answer. There exist, however, certain possibilities of such an extension. In this connection, it is of greatest interest to generalize the method to twisted (first of all slightly twisted) jets, the consideration of a weak pressure field in complex jet flows and the calculation of jets in a limited space. It is possible that in this case, the heat-conduction-type equation must be written for the sum of the dynamic and the static pressure (or the total pressure). The attempts of such a generalization of the method known to the authors are still insufficient in order to judge their effectiveness and reasonableness. In some cases (semilimited jets, antiparallel jets, etc.) it is certainly necessary to complete the empirical material in connection with the prob-

able renunciation of the invariance of the transverse coordinate.

Let us still mention a well-known advantage of the method which is connected with its applicability to, e.g., the flow of supersonic (partly not yet calculated) jets. The question is that the transition from the physical plane of the flow to the mathematical plane of (ξ, η) may be carried out beginning with an arbitrary jet cross section (which need not be the orifice cross section). For example, with a noncalculated efflux, the "initial" section ($x = x_0, y$ and correspondingly $\xi = 0, \eta$) is naturally allowed to coincide with the (virtually isobaric) end cross section of the gasdynamic part of the jet. The same holds true for other cases where the initial part of the jet cannot be taken into account. As an example, we may mention the flow with the junction of two (or several) parallel jets which intersect at a certain angle, etc. In all these cases which are important in practice an effective application of the method of the equivalent problem of the theory of thermal conductivity can be made possible by an appropriate choice of the mathematical origin of coordinates, the experimental function $\xi(x)$ and the initial distribution (e.g., of ρu^2) in the chosen section, which is basic for the calculation. Since for the solution of the heatconduction equation we have effective methods of calculation (and auxiliary tables) at our disposal, it is relatively simple to realize with the help of the initial distribution of the function investigated.

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11, 21, 22, 34, 35, 63, 65, 69, 114, 115, 142, 144, 149, 150, 153, 158, 188, 193, 199, 227, 228, 264, 281, 282, 283, 284, 321, 322.

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[Footnotes]

329 For $\rho u(u - u_\infty)$ i.e., for a jet in a parallel flow.

336 In the general case, these two terms are quantities of one and the same order of magnitude since in the estimation usual in the boundary layer theory the quantity ξ must be taken to be of the order of δ^2 which results from Eq. (14.5).

336 See also the article by Z.B. Sakinov and Zh.D. Temirbayev in the collection [154].

Chapter 15

CALCULATIONS AND COMPARISON WITH THE EXPERIMENT

15.1. SIMILARITY OF THE DISTRIBUTIONS OF A LAMINAR FLOW

In this chapter, we shall compare the results of a calculation carried out in continuing the considerations discussed in the previous chapters with various experimental materials on turbulent flows of fluid and gas. We shall do this first for the case of self-similar flows, in order to obtain an experimental verification of the assumption on the similarity of the ρu^2 distributions. The principal aim of the comparison between the results of a calculation and the experiment is, however, to study the applicability of the method of the equivalent problem of the theory of thermal conductivity to various turbulent jet flows: plane and axisymmetric jets of finite dimensions expanding in a quiescent medium or a parallel flow, with various forms of the initial velocity distribution, finally, with equal and essentially different values of the densities of the gas in the jet and that in the surrounding medium. Such a relatively broad comparison with experiment proves to be expedient in order to study the peculiarities of the method and to establish (in a preliminary plan) the limits of its applicability as well. In connection with this, with the experimental data given below, we often consider one and the same problem on the basis of experimental results of two or several authors.

The assumption of universality of the distributions of dynamic pressure ρu^2, in turbulent jets of compressible gas permits, as shown

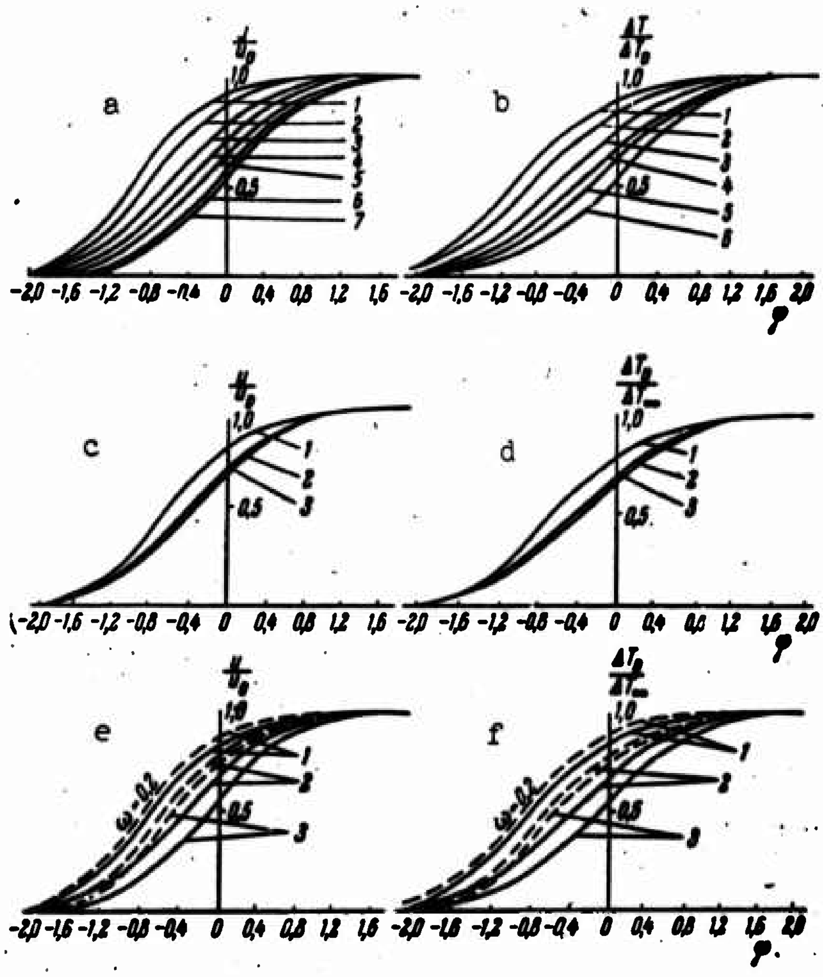

Fig. 15.1 Distributions of the values of velocity, surplus static temperature and total temperature for the edge of a gas jet ($Pr_T = 0.75$). a) Velocity; b) temperature ($M \simeq 0$);

№	1	2	3	4	5	6	7
$\omega_0 = \rho_{00}/\rho_0$	0.1	0.2	0.5	1	2	5	10

c) velocity; d) stagnation temperature $\left(\omega_0 = \frac{\rho_{00}}{\rho_0} = 1\right)$

№	1	2	3
M	4	1	0

e) velocity; ——— M = 1; f) stagnation temperature – – – – M = 4

№	1	2	3
$\omega_0 = \rho_{00}/\rho_0$	0.2	1.0	5.0

above, the final solution of calculations of the distributions of velocity, temperature, etc.

Considered from a qualitative point of view, the results of such a calculation yield a physically correct image of the expansion of a

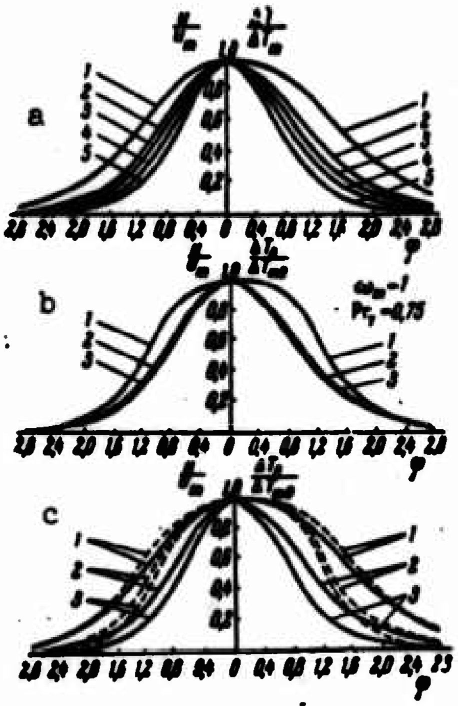

Fig. 15.2. Distributions of values of velocity and surplus (static and total) temperature in the cross sections of a plane gas jet ($Pr_T = 0.75$). a) ≈ 0

№	1	2	3	4	5
$\omega_m = \rho_\infty/\rho_m$	0.1	0.5	1.0	2.0	10.0

б) $\omega_m = 1$

№	1	2	3
M	4	1	0

в) — $M = 1$; – – – $M = 4$

№	1	2	3
$\omega_m = \rho_\infty/\rho_m$	0.1	1.0	10.0

gas jet in a medium of different density.

Here we must also add the inequality $Pr_T < 1$ which entails a more rapid drop of surplus heat content along the jet axis, compared to that of the dynamic pressure. In agreement with this, the effective thickness of the thermal layer exceeds the thickness of the dy amic layer.

These general properties of turbulent gas jets already mentioned previously become particularly clear when applied to the self-similar sections of jet flows. A great number of theoretical examples for the initial and the fundamental sections of jets is contained in a number of papers in which the ρu^2 similarity method has been used [114, 193, etc.].

By way of example, Figs. 15.1-15.3 show the corresponding pressure and temperature distributions in jet cross sections for various efflux conditions (described in the text to the figures). They characterize the influence of the two compressibility parameters: the ratio of the gas densities in the medium and a low-velocity jet ($\omega_0 = \frac{\rho_\infty}{\rho_0}$ for the edge of the jet and $\omega_m = \frac{\rho_\infty}{\rho_m}$ for plane and axisymmetric scurce jets) and the initial value of Mach's number M in a jet of high velocity. The value of the turbulent Prandtl number was assumed to be equal in all cases and given by $Pr_T = 0.75$.

These graphs have been obtained by calculations based on the assumption of universality of the distributions of ρu^2 and $\rho u c_p \Delta T$ (or $\rho u c_p \Delta T_0$ for high velocities) in the jet cross sections. The solutions of the corresponding self-similar problems for the case of ρ = const obtained by the method of the asymptotic layer are considered to be the initial solutions.

The calculations carried out are limited by the values $\omega_m = 10$ or 0.1. The first of them coincides virtually with the limiting case of $\omega_m = \infty$, for which

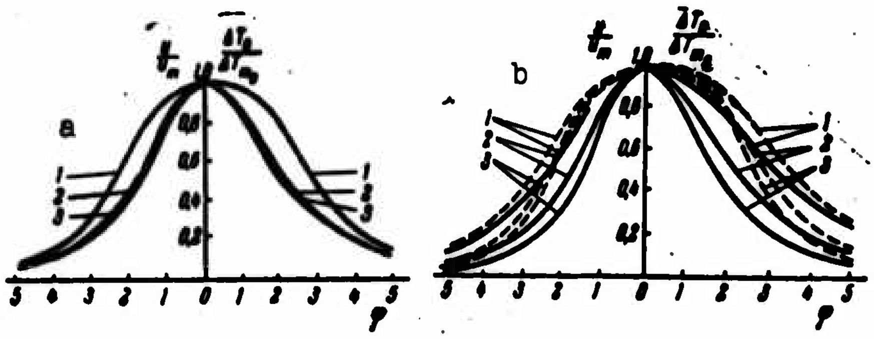

Fig. 15.3. Distributions of the values of velocity and stagnation temperature in the cross sections of an axisymmetric jet ($Pr_T = 0.75$). a) $\omega_m = 1$;

№	1	2	3
M	4	1	0

№	1	2	3
ω_m	0.1	1.0	10.0

b) —— M = 1; - - - M = 4.

$$\left(\frac{u}{u_m}\right)_{\omega_m=\infty} = \left(\frac{u}{u_m}\right)^{1+Pr_T}_{\rho=\mathrm{const}} \text{ and } \left(\frac{\Delta T}{\Delta T_m}\right)_{\omega_m=\infty} = \left(\frac{\Delta T}{\Delta T_m}\right)^{2}_{\rho=\mathrm{const}}$$

for plane and round source jets. In the second limiting case of ($\omega \to 0$)

$$\left(\frac{u}{u_m}\right)_{\omega_m=0} = \left(\frac{u}{u_m}\right)^{1-Pr_T}_{\rho=\mathrm{const}}, \text{ т. е. } \left(\frac{u}{u_m}\right)_{\omega_m=0} = \sqrt[4]{\left(\frac{u}{u_m}\right)_{\rho=\mathrm{const}}}$$

with

$$Pr_T = 0{,}75;$$

where again $\left(\frac{\Delta T}{\Delta T_m}\right)_{\omega_m=0} = 1$ for plane and round source jets. Analogous formulas for u/u_0 and $\frac{\Delta T}{\Delta T_0}$ with $\omega_0 = 0$, or $\omega_0 = \infty$ refer to the jet axis.

Recall that the comparison of the velocity and temperature distributions shown in Figs. 15.1-15.3 for various efflux conditions is conventional. As already mentioned, the curves of u/u_m or $\frac{\Delta T}{\Delta T_m}$ as functions of the coordinate φ which coincide in any of these graphs, refer actually to different cross sections: if $\omega_m > 1$ the cross section is closer to the nozzle, if $\omega_m < 1$ it is farther away from it. Therefore (see Chapter 13) they do not characterize directly the effective thick-

nesses of the jet. A more complete comparison, taking into account the variation of u_m and ΔT_m along the jet for self-similar source jets would be in connection with additional conventions which are unnecessary when the calculations are based on the method of the equivalent problem of the theory of thermal conductivity.

This remark has general character, the effective thicknesses of the jet (with respect to the dynamic pressure and, in particular, with respect to velocity or temperature) can be determined in a calculation of self-similar jets and from experimental results with insufficient accuracy. This is, first of all, due to the influence of the initial conditions: the nonuniformity of the distributions, the intensity of pulsations, the poor approximation in the case of the efflux of supersonic jets, etc. The results of this kind of calculations are therefore unsuitable for comparisons, both with one another and with experiment and are here not considered.

A great number of examples of experimental proofs of the similarity of the ρu^2 distributions and also, but not so many, of the $\rho u c_p \Delta T$ distributions are given in the papers [58, 96, 98, 114, 186, 193] chiefly for the efflux of hot jet into the atmosphere of a colder gas, the isothermal efflux of a light gas in a heavier atmosphere (or vice versa) and finally the efflux of fast gas jets (with $M \leq 1$). The similarity of the ρu^2 distributions in the case of supersonic gas jets (in the shock-wave-free part of the flow) was ascertained by many researchers (I.P. Ginzburg and his coworkers, et al.), for steam jets it was shown in paper [33], etc.

In the past years, the researchers' attention has been attracted by jets of very high temperatures obtained with the efflux of gas from plasma or other burners. In this range of temperature values with which we are concerned in this case, i.e., from several thousand to several

ten thousand degrees of a so-called low-temperature plasma [11, 31, etc.], the physical phenomena arising in the gas jet are very complex. This is, in fact, the case of a complex, multi-component mixture in which dissociation, ionization and recombination processes take place and where gas radiation etc. may play a certain part. Under the conditions of a jet motion these processes have hardly been studied experimentally.

In this connection, the first investigations of high-temperature jets carried out by G.N. Abramovich and coworkers [11, 31, 82, 83] deserve great attention. Some general results obtained in them are indubitably of interest in both principle and practice. The point is, first of all, the possibility of obtaining an approximate description of the process by way of choosing an equation of state for the pseudo-one-component gas in the form of a power function linking the enthalpy and the density [82, 83]

$$\rho = \frac{A}{i^n},$$

where A and n are empirical constants which depend on the nature of the gas.

The calculations carried out showed that Taylor's method of calculation was not suitable as it yielded physically incorrect results (non-zero stress of friction at one of the surfaces of the boundary layer of finite thickness, etc.). This conclusion agrees with the remarks on the contradictions connected with the use of Taylor's method, which we discussed already above.

Using the same mathematical method of the layer of finite thickness, the authors of [31, 82, 83] introduced different mixing lengths (dynamic and thermal) and took their ratio as an empirical constant ($Pr_T = 0.5$). It can be assumed that it would be somewhat better to use in analogous calculations the method of the asymptotic layer and a value of

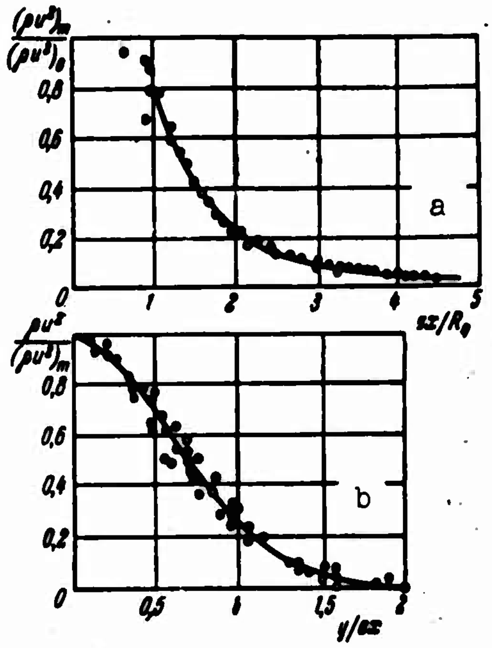

Fig. 15.4. Universal graphs of variation of the dynamic pressure along the jet axis (a) and in a cross section (b) of an axisymmetric gas jet $(0.65 < \omega_0 < 14.5$ data from [114]).

$\mathrm{Pr}_T \approx 0.7 \div 0.8$.

It is also interesting that V.A. Golubev' data [82, 83] prove the presence of a similarity of the ρu^2 distributions in the boundary layer at the edge of a plane high-temperature jet. This result was the final result obtained in a calculation which was based on quite different assumptions (boundary layer of finite thickness, accounting for the pulsations of the composite quantity ρu according to E. Van Drist [41] etc.). We see that the assumption of an approximate similarity of the ρu^2 distributions for the self-similar sections of low-temperature plasma jets agree satisfactorily with the experiment. In any case, the supposition of similarity of the velocity under these conditions is rather far from reality. As regards the laws of variation of the flux density of surplus enthalpy in plasma jets, the investigation into this

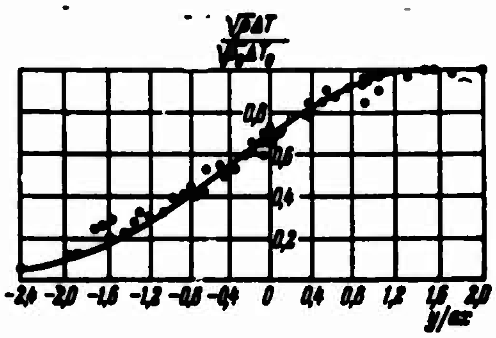

Fig. 15.5. Distribution of the value $\Delta\tau = \sqrt{\bar{\rho}}\, c_p T$, for the edge of a noniso-thermal (data from [114], $0 < m_0 < 5.4$):

problem has only begun. Note that the distribution of the quantity $\rho u \Delta i$ may be influenced noticeably by radiation. It is the merit of the papers we referred to that they indicated the possibility of a transition to a new field of investigations of mathematical methods and representations of the theory of turbulent jets, which justified themselves under "usual" conditions.

As regards the comparison with experiment of the mathematical results obtained under the supposition of similarity of the distributions of ρu^2 and $\rho u c_p \Delta T$, we give, in addition to Fig. 9.5, the graphs of Figs. 15.4-15.5, referring to the efflux of a heated jet [114, 187].

We see from the figures given that, with an appropriate choice of the constant a, the distributions of ρu^2 and $\rho u c_p \Delta T$ are approximately similar.

In a series of cases [11] it is possible in a good approximation to achieve coincidence with a virtually universal distribution, of the primary velocity distributions, again by way of choosing an empirical constant. Thus, in many cases, both suppositions (similarity of the ρu^2 distributions or similarity of the u-distributions) may be used in en-

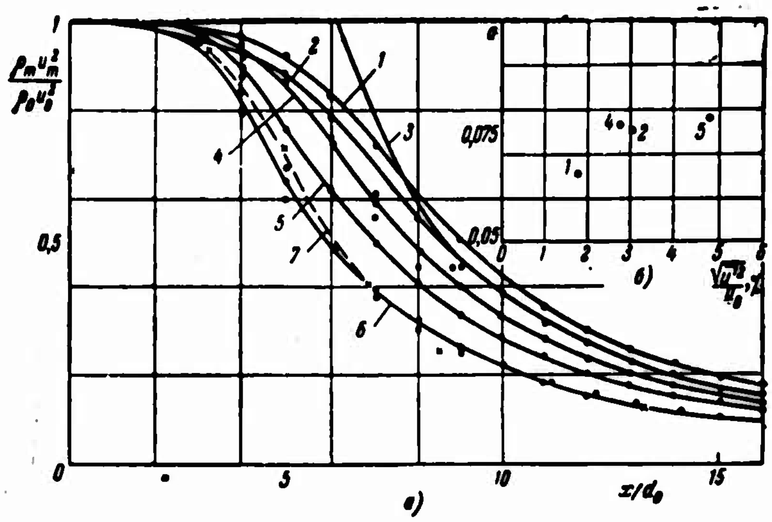

Fig. 15.6. Variation of ρu^2 along the axis of a round jet. 1) With a fine grid arranged at the nozzle orifice; 2) efflux from tube ("one-seventh" distribution); 3) calculation after G.N. Abramovich $(x_0 = 0, c = 0,22;$ see [11]); 4) efflux from "drawn-in" nozzle; 5) turbulizer arranged in the wide part of the nozzle; 6) efflux of hot jet ($\omega_0 = 3.6$); 7) hot gas jet $T_0 \approx 4000°$ K, $\omega_0 \approx 14$; see [31]).

Note: In the inset in the upper right corner the values of the empirical constant a are given as depending on the intensity of the pulsations.

gineering calcudations, though, as shown above, the procedure based on the similarity of ρu^2 similarity) is more convenient also from the point of view of practice. Thus, for example, in the initial section of parallel nonisothermal jets, coincidence of the curves for different values of the ratio ρu^2 is achieved by means of choosing one constant a for each of the curves. This is possible since, as shown by experiment, [96, 150, etc.], the quantity $\frac{U - U_\infty}{U_0 - U_\infty} \approx 0{,}7$ with $y = 0$ (in this case $U = \sqrt{\rho u}$). Unlike this [217, 218], the construction of a universal velocity distribution requires the introduction of two empirical constants for each of the curves which are necessary for the coincidence of the different values of $\frac{u - u_\infty}{u_0 - u_\infty}$ with $y = 0$ firstly, and for the usual transformation of the transverse coordinate, secondly.

It is essential that the construction of universal distributions

of ρu^2 requires the introduction of the empirical constant a whose value cannot be predicted beforehand. It can only be shown that the numerical value of a is influenced considerably by a series of factors. Under otherwise equal conditions the initial degree of turbulence is of principal significance (under conditions where in the outlet section of the nozzle the transverse component of the mean velocity is virtually vanishing). The significance of this factor is great not only for turbulent gas jets but also for turbulent jets of incompressible fluid. A characteristic example is shown in Fig. 15.6, namely the curves of variation of $(\rho u^2)_m$ along the axis of a submerged jet. The upper three curves refer to an isothermal jet with uniform initial velocity distribution but different conditions of initial turbulence. The two lower curves refer to nonisothermal jets with different heating (about 1200°K and 4500-500°K). Details of the experiment are given in the text to the figure.

As we see from the latter, the influence of the initial conditions on the intensity of attenuation of the jet (according to the value of the constant a) is very high*. A still greater influence on the "scatter" of the jet is exerted by the presence of a transverse velocity component in the initial section of the jet. Moreover, a noticeable effect is also exerted by a nonuniformity of the initial distributions of ρu^2, $\rho u c_p \Delta T$ etc. (velocity and temperature for incompressible fluid) which has been taken into account in a series of papers [11, 178, 190, etc.] by introducing additional coefficients.

Under the conditions of a nonrated efflux the value of the constant a is strongly influenced also by the degree of nonratedness of the efflux, i.e., the gasdynamical history of the turbulent jet. As regards the immediate influence of the density ratio ω on the constant a, it is disputable for an evolved turbulent motion. It is obvious that the

indirect influence of this parameter exerted via the initial level of turbulence is considerable. In particular, a heating of the gas by a flame raises the level of pulsations essentially [64] but the difference in the values of the constant a are very small in a wide range of the heating degree, although a transition from an isothermal jet to a jet consisting of combustion products (Fig. 15.6) is accompanied by a steep increase of the value of the constant a. As will be shown below (see Section 15.5) with a slightly raised initial level of turbulence the differences in the behavior of isothermal and nonisothermal jets (heated by flame) are virtually vanishing.

Summing up we see that the problem of the influence of the various factors on the value of the empirical constant a is investigated insufficiently, in spite of the huge experimental material on turbulent jets. In essential, only the first attempts are made to exert an active influence on this flow characteristic which has a complex and not at all unambiguous nature. On the basis of general considerations one may prove (and this is in agreement with many experiements) that, with an increased initial level of turbulence, the differences in the values of a become smaller. It is therefore valuable in practice, with "usual" conditions, to work with the universal constant suggested by G.N. Abramovich [11] (the corresponding curve has been plotted in the same Fig. 15.6).

It would be of great interest to compare experiment and calculated formulas based on the similarity of ρu^2 for velocity distributions in the limiting cases of an infinitely lare or infinitesimal value of the density ratio of the gases in the jet and the surrounding medium. Unfortunately, reliable experimental data are not available for such a comparison. Note, however, that the material given in papers [11, 80] on the efflux of water in air and air in water agree qualitatively with

what has been said above.

15.2. A PLANE JET OF INCOMPRESSIBLE FLUID

Let us consider some examples of plane turbulent jet flows of incompressible fluid which may illustrate the calculation according to the method of the equivalent problem of the theory of thermal conductivity, and compare the results with experiments. By way of example we consider three types of flow described in a paper by B.P. Ustimenko [199] and, investigated in paper [322], the expansion of a plane jet in a parallel flow.

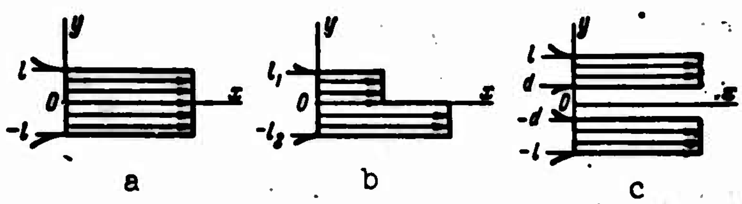

Fig. 15.7. Schematic representation of turbulent jets. a) Jet of finite dimensions; b) complex initial distribution; c) two jets.

The solution of a heatconduction-type equation

$$\frac{\partial}{\partial \xi}(\rho u^2) = \frac{\partial^2}{\partial y^2}(\rho u^2)$$

with the initial condition

$$\rho u^2 = f(y) \quad \text{with} \quad \xi = 0$$

and the boundary conditions $\frac{\partial \rho u^2}{\partial y} = 0$ with $y = 0$ and $\rho u^2 \to 0$ with $y \to \infty$, where $\xi = \xi(x)$, can be represented in an analytical form by [188]

$$\rho u^2 = \frac{1}{2\sqrt{\pi\xi}} \int_{-\infty}^{+\infty} f(\eta) \exp\left[-\frac{(\eta - y)^2}{4\xi}\right] d\eta.$$

Let us apply this well-known solution to the flows shown schematically in Fig. 15.7a for a jet with uniform initial velocity and temperature distributions and in Fig. 15.7b a jet with an initial distribution of the form of a "step." In the first case the solution reads

$$\frac{u}{u_0} = \frac{1}{\sqrt{2}}\left[\operatorname{erf}\left(\frac{\bar{y}+1}{2\sqrt{\bar{\xi}}}\right) - \operatorname{erf}\left(\frac{\bar{y}-1}{2\sqrt{\bar{\xi}}}\right)\right]^{1/2},$$

where $\bar{y} = \frac{y}{l}$, $\bar{\xi} = \frac{\xi}{l^2}$, l is the half width of the slit; an analogous form possesses the solution of the thermal problem (also in the case of a uniform initial distribution of the temperature difference) when written in terms of the reduced coordinate ξ instead of ξ_T.

In the second case

$$\frac{u}{u_0} = \frac{1}{\sqrt{2}}\left[(m^2-1)\operatorname{erf}\left(\frac{\bar{y}}{2\sqrt{\bar{\xi}}}\right) - m^2 \operatorname{erf}\left(\frac{\bar{y}-1}{2\sqrt{\bar{\xi}}}\right) + \operatorname{erf}\left(\frac{\bar{y}+1}{2\sqrt{\bar{\xi}}}\right)\right]^{1/2},$$

where $m = \frac{u_1}{u_0}$.

Experiments were conducted (see [199]) with an air jet leaving a slit nozzle with a speed of 40-80 m/sec; in order to study the thermal problem, the air was preheated such that its temperature exceeded that of the surrounding medium by 10-15°. To study the "step" jet, experiments were made at various velocity ratios $m = \frac{u_1}{u_0}$. The relation linking the physical coordinate x and the reduced coordinate ξ (and that for x and ξ_T) the experimental and theoretical dependences of velocity and surplus temperature along the jet axis on the distance x/l and on the reduced coordinate $\bar{\xi} = \frac{\xi}{l^2}$ were compared.

The data shown in Figs. 15.8 and 15.9 apply to the flow patterns a) and b) of Fig. 15.7 and prove the satisfactory agreement between the theoretical and experimental velocity and temperature distributions in the jet cross sections. Just as in other cases of experimental data processing according to the method of the equivalent problem, this agreement is achieved by a combination of the coordinates $\sqrt{\bar{\xi}}$ and x/l, i.e., such a deformation of the theoretical coordinate which allows the theoretical and experimental distributions to coincide on the flow axis.

In this way, the method of the equivalent problem permits in a good approximation a continuous deformation of the complex initial dis-

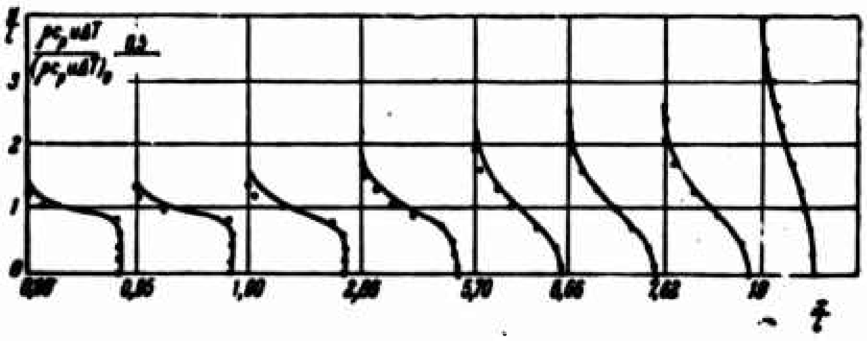

Fig. 15.8. Distribution of surplus heat content in the cross sections of a plane turbulent jet. ——calculation; oooo experiments [199].

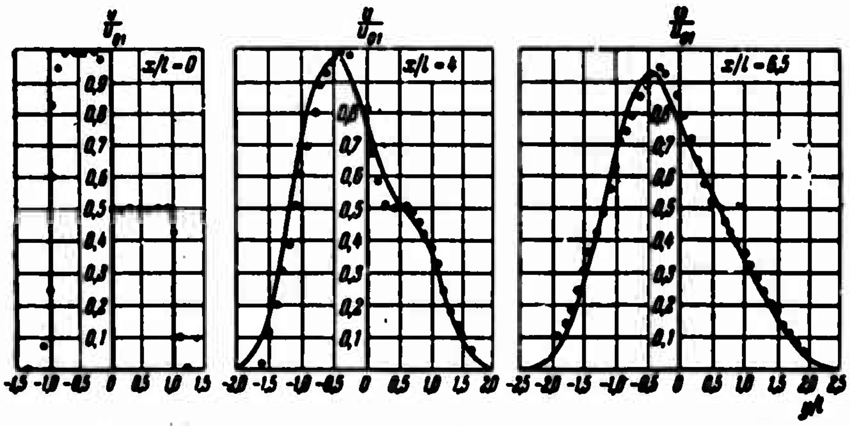

Fig. 15.9. Comparison of the theoretical velocity distributions with experimental data for a complex jet ($m = 2$). ——calculated; oooo experiment [199].

tribution. We should also note that in the first of the cases considered the ratio $\frac{\varepsilon}{\varepsilon_T}$ is almost constant and equal to 0.8. As already mentioned before, in the method of the equivalent problem this ratio plays the part of the peculiar turbulent Prandtl number.

At great distances from the nozzle the velocity and temperature distributions correspond to those of a flow produced by a source jet. In this case, the transverse distribution will be universal; a solution of the form

$$\frac{u}{u_m} = \exp\left(-\frac{1}{2}\varphi^2\right),$$

will correspond to it; here $\varphi = \frac{y}{ax}$. The value of the constant a is chosen from experimental data.

It is also worth mentioning that in the paper referred to, it has been tried to apply this method to the problem schematically illustrated in Fig. 15.7c. In this case, however, a coordinate transformation $\xi = \xi(x), \eta = y$ with the origin at the point $x = 0$ only yields qualitative agreement at short distances from the nozzle [199]. This result was to be expected from general considerations since the "adherence" of the jet observed in real flows causes an inevitable deformation not only of the longitudinal but also of the transverse coordinate. A detailed experimental investigation of such a flow with respect to the distribution of mean velocity, pressure and pulsations given in papers [266,291, 292] explains the inapplicability of the simplest forms of variables transformations to such a flow.

The example given is characteristic, on the one hand, as it shows the limitedness of the method of the equivalent problem (in its simplest form: $\xi = \xi(x), \eta \cong y$).

On the other hand, if one uses the same method (in the same form) not for the whole jet flow produced by two parallel jets but only for the section after their fusion (i.e., the theoretical origin of coordinates $\xi = 0$ is about two nozzle diameters away from the nozzle), the agreement between calculation and experiment will be quite satisfactory.

In this case, it is, of course, the distribution in the chosen cross section at $\xi = 0$. which is considered to be the "initial" one.

In this way, the above example illustrates partly but nevertheless effectively the possibility of applying this method to a very complex

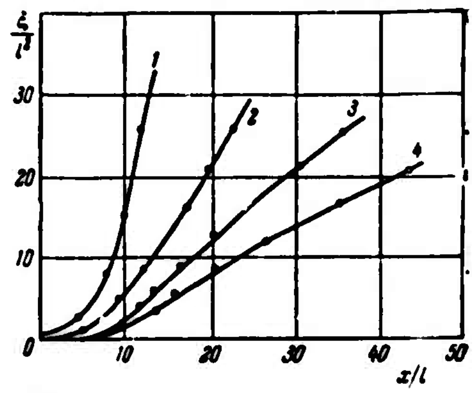

Fig. 15.10. Dependence of the effective coordinate on the distance for a plane jet expanding in a parallel flow.

1 — $m = (u_{стр}/u_{пот}) = 0$; 2 — $m = 0,33$; 3 — $m = 0,5$; 4 — $m = 0,66$

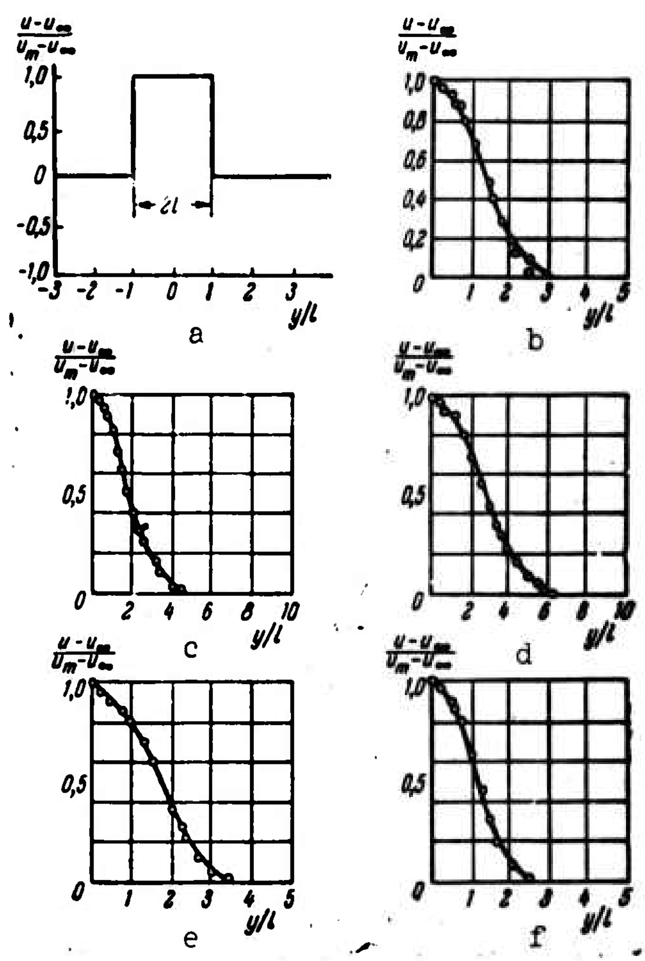

Fig. 15.11. Velocity distributions in the cross sections of a plane jet expanding in a parallel flow. ——— calculations with the help of a hydraulic integrator [34, 35], oooo experiments [244]; a) initial velocity distribution; б) $m = 0,5$; $x/l = 15$; в) 0,5; 32; г) 0,5; 60; д) 0,33. 15; е) 0,67; 15.

jet problem.

In the paper of a group of authors [322] a calculation after H. Reichardt was applied to the processing of experimental data on the expansion of a plane turbulent jet of incompressible fluid in a uniform parallel flow. The same problem was studied carefully by V.G. Bespalova with the help of a hydraulic integrator [34, 35]. Without entering into details and giving the solution we restrict ourselves to the remark that for the difference $\rho u^2 - (\rho u^2)_\infty$ the equivalent equation and its solution remain the same as in the case of the efflux of a jet into a quiescent medium. The connection between the reduced coordinate ξ of the linearized space and the real coordinate x (and the velocity ratio $m = \frac{u_0}{u_\infty}$) for each of the values) will be different.

Figure 15.10 shows $\frac{\xi}{l}$ as a function of x/l for four values of m obtained from a comparison of the theoretical solution with experiments described in paper [322]. In Fig. 15.11, we see that the distributions calculated (with the help of a hydraulic integrator [34]) for the cross sections of a plane jet expanding in 1 medium at rest or in a parallel flow agree satisfactorily with experimental data [244, 322].

Note that the solutions (and also the experimental data) for a flow away from the nozzle, where the difference between the velocity values in the jet and the surrounding parallel flow are small, is in good agreement with the results of a calculation according to the method of small perturbation.

The examples given here are sufficient to judge the effectiveness of the method of the equivalent problem of the theory of thermal conductivity when applied to calculating plane turbulent jets.

15.3. AN AXISYMMETRIC JET OF FINITE DIMENSIONS

We shall now extend the conclusions obtained above to a free turbulent axisymmetric jet of incompressible fluid issuing from a nozzle

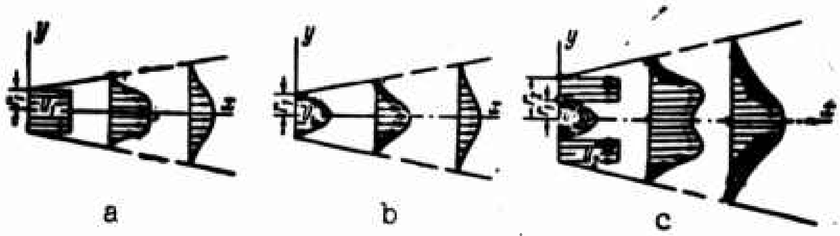

Fig. 15.12. Schematic representation of a free turbulent jet. a) Case of constant initial velocity; b) efflux from long tube; c) efflux from concentric nozzles.

of finite dimensions.

As in the previous section, our main interest is devoted to the investigation of the continuous deformation of the velocity (and temperature) destribution. In this field, the problem cannot be solved by the usual semiempirical methods of jet calculation.

Concrete examples of flows taken from the experiments by I.B. Palatnik [149, 150] are shown in Fig. 15.12 a-c. They correspond to the following problems: efflux of a jet with uniform initial velocity distribution and an evolved "tubular" (according to the law of "one seventh") initial velocity distribution with a uniform temperature distribution in both cases, and also the efflux from two concentric round nozzles (experiments by S.B. Stark [178] and I.B. Palatnik [150]).

The solution of the equivalent problem of thermal conductivity with the corresponding boundary conditions for the above forms of flow was obtained by combining the analytical solution (for the axial values of velocity and temperature) and the solution obtained by means of a hydrodynamic integrator (transverse distributions). The first of them (on the jet axis) was used in order to establish the sought connection between the reduced coordinate ξ and the nozzle distance x/d.

We know a solution [188] of the heatconduction-type equations for ρu^2

$$\frac{\partial}{\partial \xi}(\rho u^2) = \frac{1}{y}\frac{\partial}{\partial y}\left[y\frac{\partial}{\partial y}(\rho u^2)\right]$$

and an analogous one for $\rho u c_p \Delta T$:

$$\frac{\partial}{\partial \xi_T}(\rho u c_p \Delta T) = \frac{1}{y}\frac{\partial}{\partial y}\left[y\frac{\partial}{\partial y}(\rho u c_p \Delta T)\right]$$

with boundary conditions of the form

$$\begin{array}{lll} u = u_1(y), & \Delta T = \Delta T_1(y) & \text{with } x = 0 \text{ and } 0 < y < r_1, \\ u = u_2(y), & \Delta T = \Delta T_2(y) & \text{with } x = 0 \text{ and } r_1 < y < r_2, \\ \dfrac{\partial u}{\partial y} = \dfrac{\partial \Delta T}{\partial y} = 0 & & \text{with } x > 0 \text{ and } y = 0, \\ u = \Delta T = \dfrac{\partial u}{\partial y} = \dfrac{\partial \Delta T}{\partial y} = 0 & & \text{with } x > 0 \text{ and } y \to \infty, \end{array}$$

For example, for the momentum flux density or the flux density of the surplus heat content, we can write

$$z = 2\pi \int_0^\infty \left[\int_0^1 z_1(\alpha) J_0(\lambda\alpha)\,\alpha\, d\alpha + \int_1^{r_2/r_1} z_2(\alpha) J_0(\lambda\alpha)\,\alpha\, d\alpha\right] \exp(-\lambda^2 \xi) J_0(\lambda \bar{y})\,\lambda\, d\lambda,$$

where $z = \frac{\rho u^2}{(\rho u^2)_{hm}}$ or $z = \frac{\rho u c_p \Delta T}{(\rho u c_p \Delta T)_{hm}}$
(in this case in the expression for z the variable ξ must be replaced by ξ_T), $z_1(y)$ and $z_2(y)$ denotes the function of the initial momentum flux density distribution or the flux density of surplus heat content, which are given by the boundary conditions; $J_0(\lambda\alpha)$ is a Bessel function of the first kind and zero order; the integration with respect to α is carried out over the nozzle area; $\bar{y} = y/r_1$.

The functions obtained when comparing the theoretical and experimental curves of variation of ρu^2 (and $\rho u c_p \Delta T$) along the jet axis are shown in Figs. 15.13 and 15.14. In Figs. 15.15 and 15.16 the correlation between the coordinates ξ and x are shown. It is worth mentioning that the function $\xi(x)$ obtained for a jet issuing from concentric nozzles is

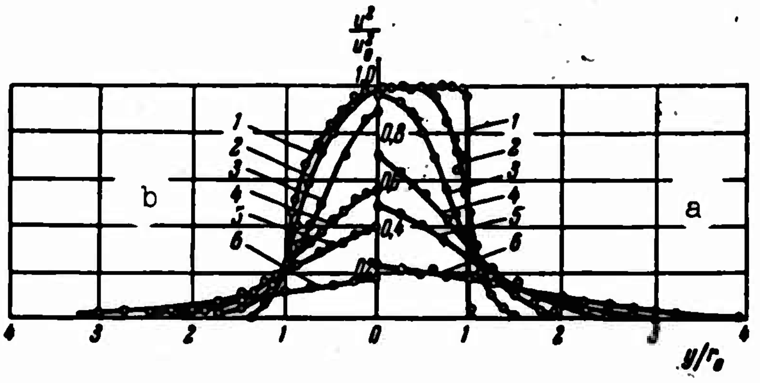

Fig. 15.13. Momentum flux density distributions in the cross sections of a round jet [149]. a) According to the diagram of Fig. 15.12,a; b) according to the diagram of Fig. 15.12,b.

№	1	2	3	4	5	6
x/d	0	2	5	8	10	15

virtually the same even if the ratios of the nozzle diameters and the velocity values are different.

In the figures given the experimental transverse distributions of velocity and temperature in the flows investigated are compared with the theoretical results obtained by the method of the equivalent problem (solid lines in all figures). In all cases, the coincidence must be considered to be fully satisfactory.

It should be noted in particular that the investigation into the efflux of jets from concentric nozzles is of immediate interest in practice, for combustion and furnace engineering and the like. The possibility of a detailed calculation of velocity and temperature distributions on the basis of the modest experimental data (applying to the jet axis) therefore indicates the promising aspects of an application of the method of the equivalent problem to investigations of this type of flows. In this connection, V.A. Adamovich's results [21, 22] of investigations into the aerodynamics of metallurgical furnace burners. This author studied the initial velocity and temperature distributions

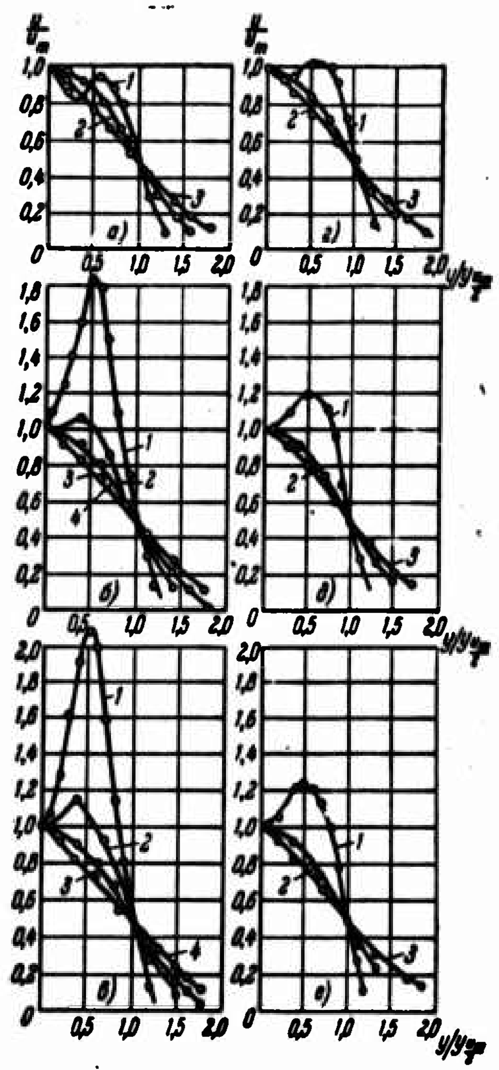

Fig. 15.14. Velocity profiles in cross sections of a complex jet (flow according to the diagram of Fig. 15.12, c) —— calculation by the integrator [149]; oooo experiments [178].

№	1	2	3	4
x/d	5	10	15	25

(with ρ = const) of air jets issuing from coaxial tubes in experiments which were more complicated than those mentioned above. V.A. Arutyunov simulated the mixing of three flows: gas jets issuing from two coaxial tubes (central and peripheral) and the parallel flow surrounding the first ones. V.A. Arutyunov [21,22] could show that a processing of the experimental data according to the method of the equivalent problem in all cases investigated yielded positive results, obtained by means of a single experimental function $\xi = \xi(x)$(and for the flow away from the nozzle by a single empirical constant) which resulted in a good agreement between calculated and experimental velocity and temperature distributions. It was shown that in practice, within a well-known range, one can use a single general function $\xi(x)$. Finally, as also in other papers, the ratio ξ/ξ_T proved to be equal to about 0.8.

It is obvious that for direct jet burners with arbitrary initial velocity distributions, the application of the method of the equivalent problem is quite rational.

The data given in this section refer to jets of incompressible fluid. As regards gas jets, in the case of the efflux of a jet in a quiescent medium, a particular case of its expansion in a parallel flow, detailed data will be given in Section 15.5 devoted to the problem of a gas jet in a parallel flow. It is expedient only to consider one case of jet issuing into a quiescent medium as an exception. This is the case of a gas jet (hydrogen plasma) of a very high temperature.

Figure 15.17,*a* shows experimental data by N.G. Zabudkina on the dynamic pressure distributions for different distances from the nozzle, obtained for such a jet with an initial value of the temperature of $T_0 \cong 5200°$ K and a velocity of $u_0 \cong 620$ m/sec. In this figure, the experimental data are represented by empty circles. The solid lines represent the theoretical functions obtained by the method of the equiva-

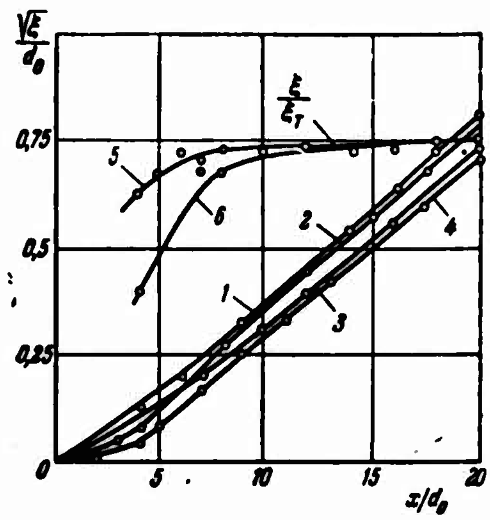

Fig. 15.15 Depen..ence of $\bar{\xi}$ on $\bar{x}$ for jet flows accor ing to the diagrams of Fig. 15.12,*a* and *b* [149]. 1,3,5 – $\bar{\xi}$, $\bar{\xi}_T$ and ξ/ξ_T for the diagrams c Fig.15.12,*a*; 2,4,6 – the same for the diagram of Fig.15.12,*b*.

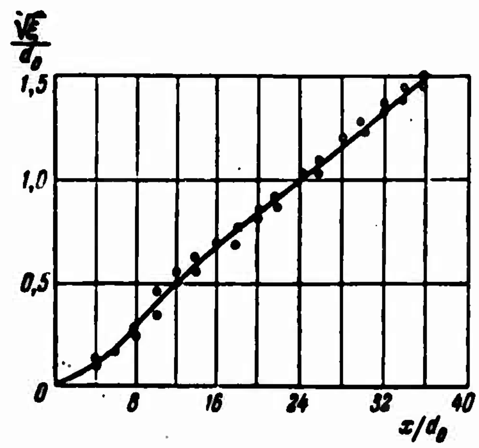

Fig.15.16. Dependence of $\bar{\xi}$ on $\bar{x}$ for the diagram of Fig. 15.12,*c* (for various diagrams of flow) [149,150].

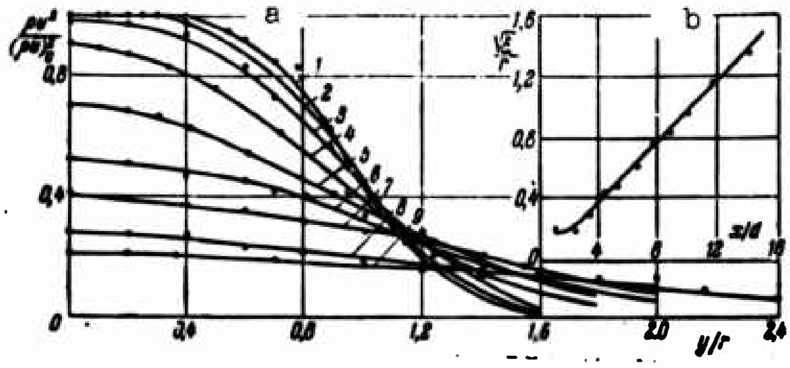

Fig. 15.17. a) Distribution of dynamic pressure in a plasma jet. (Experiments N.G. Zabudkina) oooo experiments; ——— calculation according to the method of the equivalent problem.

№	1	2	3	4	5	6	7	8	9
x/d	1	2	3	4	5	6	7	9	10

b) dependence of ξ on x.

lent problem of the theory of thermal conductivity. The "key" of the transition, in the form of the dependence of the reduced coordinate ξ on x, is shown in Fig. 15.17, *b*. As we see, with a value of the compressibility parameter $\omega = \frac{\rho_\infty}{\rho_0} = \frac{T_0}{T_\infty} \cong 14$, as also in other cases of a jet of finite dimensions, the method of the equivalent problem (with a deformation of the longitudinal coordinate alone) is well suited for a calculation of the flow. It is also characteristic that at a considerable distance from the nozzle, we can use the simple relation $\sqrt{\xi} = cx$, where $c \approx 0.05$, i.e., the order of magnitude of the constant c is precisely the same as in experiments with flame heating and a value of ω which is essentially lower.

In this way, the data by N.G. Zabudkina verify the above assumption that for low-temperature plasma jets the method of the equivalent problem remains applicable. Before we pass over to the problem of a jet in a parallel flow, the main topic of this section, we shall consider the application of this method of the equivalent problem to a three-

dimensional jet.

15.4. THE THREE-DIMENSIONAL JET

As already mentioned, the method of the equivalent problem of the theory of thermal conductivity can be used not only in order to treat two-dimensional (plane and axisymmetric) jets, but also for three-dimensional, spatial problems. In this case, the method in its simplest form which is therefore most convenient for practice, results in the seeking of an experimental connection between the physical space and a fictitious space in which the equations are linear, for the longitudinal coordinate $\xi(x)$ alone. Having obtained this relation from experiment (for example, from the distance dependence of the dynamic pressure ρu^2 along the jet axis) the other two coordinates η and ζ must be considered to be, in a first approximation, coincident with the corresponding coordinates y and z.

We are thus concerned with the transition from the system of equations of the three-dimensional boundary layer

$$\frac{\partial \rho u^2}{\partial x}+\frac{\partial \rho uv}{\partial y}+\frac{\partial \rho uw}{\partial z}=\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}, \qquad \frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}+\frac{\partial \rho w}{\partial z}=0$$

with the boundary conditions

$$\overline{\rho u^2}=\frac{\rho u^2}{\rho u_0^2}=F_0(y, z) \text{ with } x=0,$$

$$\overline{\rho u^2}\to 0 \qquad \text{with } y\to\infty,\ z\to\infty,$$

to a two-dimensional heatconduction-type equation

$$\frac{\partial \rho u^2}{\partial \xi}=\frac{\partial^2 \rho u^2}{\partial \eta^2}+\frac{\partial^2 \rho u^2}{\partial \zeta^2},$$

where $\xi=\xi(x),\ \ \eta=y,\ \ \zeta=z,$ with the a alogous boundary conditions

$$\overline{\rho u^2}=F_0(\eta, \zeta) \text{ with } \xi=0,\ \overline{\rho u^2}\to 0 \text{ with } \eta\to\infty,\ \xi\to\infty.$$

Note that the initial system of equations of the boundary layer is not closed since for the three velocity components we only have two equations. In the transition to the effective space we obtain a single equation for the determination of the fundamental longitudinal velocity

component u. As also in the initial boundary layer equations, the additional application of the continuity equation does not permit the determination of the two transverse velocity components v and w. This peculiarity of the three-dimensional boundary layer equation is well known (see, e.g., [135]). For the problem considered here it is, however, sufficient only to determine the fundamental velocity component u.

A solution to the heatconduction-type equation may be obtained by the method of summation over the sources in the form

$$\frac{\rho u^2}{(\rho u^2)_0} = \frac{1}{4\pi\xi}\int_{-b}^{+b}\int_{-c}^{+c} F_0(y, z)\exp\left[-\frac{(y-\eta)^2+(z-\zeta)^2}{4\xi}\right]d\eta\, d\zeta.$$

Remembering the comparison with the experimental data where the initial distribution is uniform, we represent the solution in the form

$$\frac{\rho u^2}{(\rho u^2)_0} = \frac{1}{4}\left[\operatorname{erf}\left(\frac{y+b}{2\sqrt{\xi}}\right)-\operatorname{erf}\left(\frac{y-b}{2\sqrt{\xi}}\right)\right]\left[\operatorname{erf}\left(\frac{z+c}{2\sqrt{\xi}}\right)-\operatorname{erf}\left(\frac{z-c}{2\sqrt{\xi}}\right)\right].$$

For (ρu^2) as a function of the coordinate ξ we have

$$\frac{(\rho u^2)_m}{(\rho u^2)_0} = \operatorname{erf}\left(\frac{b}{2\sqrt{\xi}}\right)\operatorname{erf}\left(\frac{c}{2\sqrt{\xi}}\right).$$

This expression is suitable for a comparison with experiment and the establishment of the relation between the coordinates ξ and x. Note that for large values of ξ (self-similar region of flow) this expression can be simplified and represented in the form

$$\frac{(\rho u^2)_m}{(\rho u^2)_0} \to \frac{1}{4\pi\frac{\xi}{S}} \quad \text{with} \quad \frac{\xi}{S} > 1, \text{ where } S = 4bc.$$

Let us now turn to the experiments conducted by I.B. Palatnik and D.Zh. Temirbayev [154]. In the experiments one studied the dynamic pressure dist ibution in air jets discharged from rectangular nozzles with different values of the side ratio. The quantity $\sqrt{S}$, was taken as the characteristic geometrical dimension of the nozzle, where $S = 4bc$, b and c being the sides of the rectangle along the axes y and z. Data by V.A. Turkus [195] were partly also comprised in the processing. As we

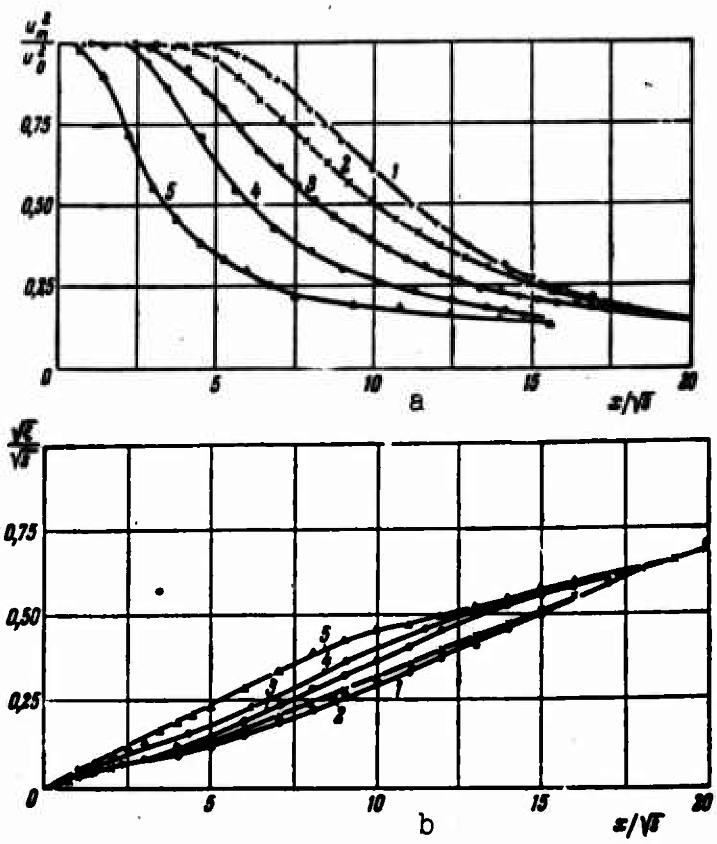

Fig. 15.18. a) Variation of u^2 along the axis of a jet issuing from a nozzle of rectangular cross section (data by I.B. Palatnik and D.Zh. Temirbayev [154]). oooo experiments; —— calculation according to the method of the equivalent problem; b) ξ as a function of x.

№	1	2	3	4	5
$n=\frac{b}{c}$	1	2	3	5	20
b мм	20	30	30	500	500

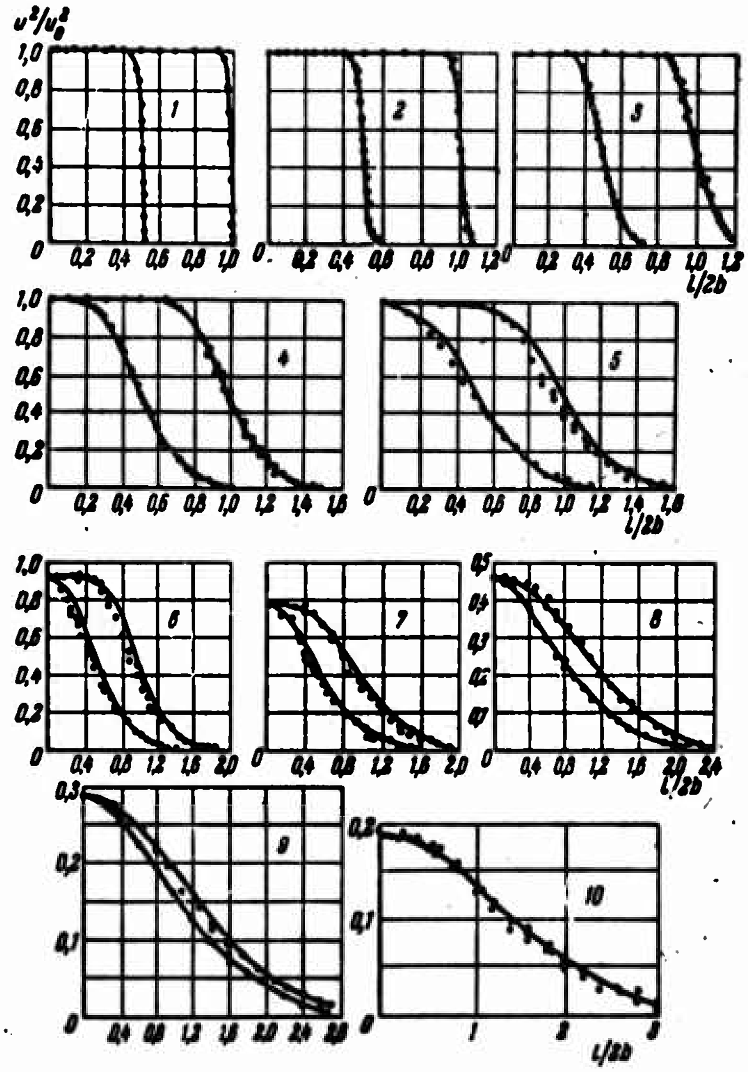

Fig. 15.19. Distribution of u^2 in a jet with rectangular nozzle cross section (n = 2 [ref]).

●●● – $l = y$, $z = 0$; ○○○ – $l = z$, $y = 0$ } ——— experiment; – – – – calculation.

№	1	2	3	4	5	6	7	8	9	10
x/b	0	1	2	4	6	8	10	15	20	40

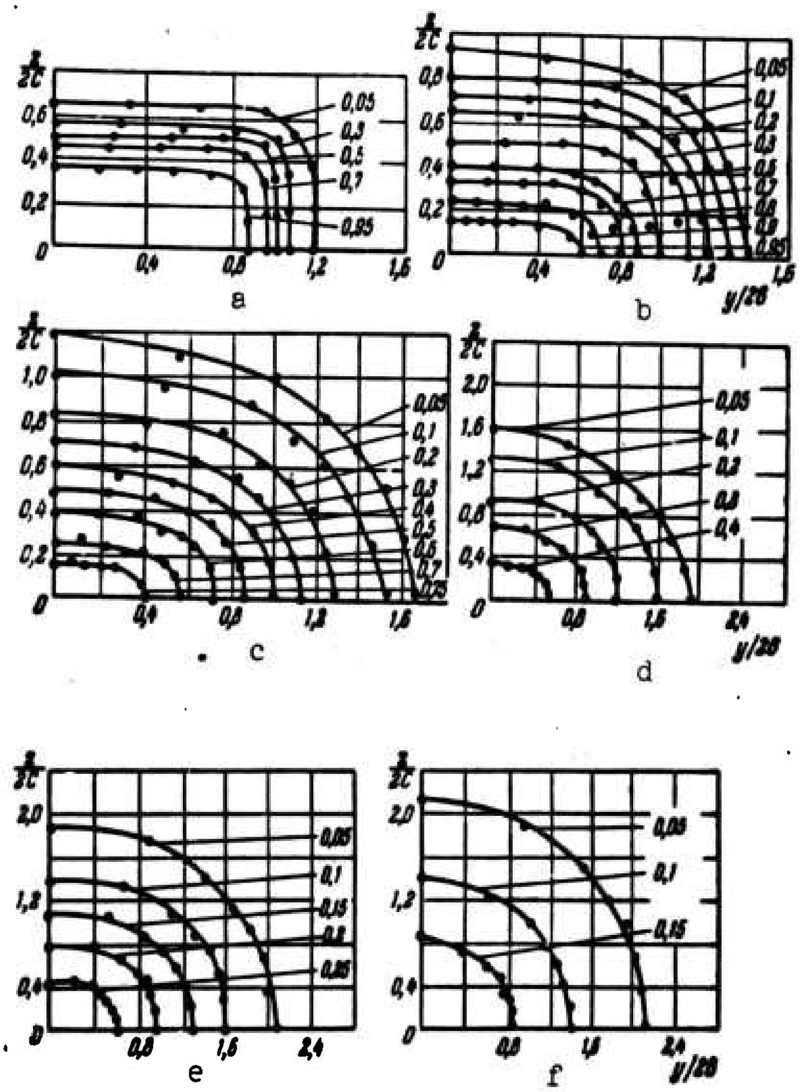

Fig. 15.20. Experimental (●●●) and calculated (——) according to the method of the equivalent problem – curves of constant value of the dynamic pressure in the cross sections of a jet issuing from a rectangular nozzle ($n = 2$ [69]).

Рис.	а	б	в	г	д	е
x/b	2	6	10	15	20	25

Note: The corresponding values of u^2/u_0^2 are given for each curve.

see from the experimental data, with $x_0/\sqrt{S} \geqslant 20$ the flow becomes similar to a self-similar flow (the value of $x_0/\sqrt{S}$ is different for nozzles of different geometry, i.e., of a different value of the side ratio $n = b/c$. Figure 15.18 shows the experimental curves of the dynamic pressure drop along the jet axis for several values of n (the nozzle dimensions are given in the text to the figures). If in the range of $x/\sqrt{S} > 1$ all curves tend to unity, which corresponds to the equivalent round jet, they differ essentially between one another. Using the connection between the coordinates ξ and x (see Fig. 15.18,b) we can compare the dynamic pressure distributions in the jet cross sections with the solution of the heatconduction equation. Such a comparison was carried out. The results for one value $n = 2$ (nozzle No. 2) are shown in Figs. 15.19 and 15.20. Analogous data were also obtained for other nozzles. In the first of these figures, we see the u^2 distributions, in the second the curves of constant value of u^2 at different distances from the nozzle. The solid lines have been drawn according to the solution of the equation (taking the relation between ξ and x into account), the dots give the experimental data. As we see from the figures, the method of the equivalent problem can be applied without any doubt to investigations of three-dimensional jets. Later on, this was proved by I.B. Palatnik in experiments with jets issuing from a nozzle with a cross-shaped orifice.

Though it is not unexpected, the result obtained is significant in principle.

15.5. AN AXISYMMETRIC GAS JET IN A PARALLEL FLOW

The problem of the expansion of an axisymmetric jet in a parallel flow (for example, the jet of combustion products from a flying jet engine) has attracted great interest. In particular, the applicability

to it of the method of the equivalent problem of the theory of thermal conductivity was verified in a series of papers [69, 149, 150, 321] in which the techniques of experiment and data processing were further developed and improved and the region of variation of the fundamental parameters of the process was enlarged. At present, we may say that this verification has been carried out rather carefully for subsonic jets issuing into a gaseous atmosphere of high density. In this case, the low density of the gas in the jet was chiefly the result of heating by flame. The variation of the compressibility parameter $\omega = \frac{\rho_\infty}{\rho_0}$ was therefore limited by the values $0 \leqslant \omega \leqslant 5$. The experimental data given below and the results of their processing according to the method of the equivalent problem are chiefly taken from detailed experiments carried out by L.P. Yarin [67, 154] and V.Ye. Karelin (115, 154]. In the first-mentioned papers (as also in previous papers, see, e.g., [150]) the dimensions of the orifice cross section of the nozzles for the jet and the parallel flow surrounding it were relatively small (inner diameter ∿20 mm, outer diameter ∿300 mm). In a later experiment by V. Ye. Karelin [69, 115] an arrangement was used with larger dimensions (tube diameters 50 and 600 mm) which permitted a higher accuracy and reliability of the measurements.

In L.P. Yarin's paper, dealing with the efflux of hot jets (initial temperature 300-1520°K with jet velocities of up to 80 m/sec and a velocity of the parallel flow of up to 20 m/sec), the influence of the parameter m, the ratio of the dynamic pressures in flow and jet (in the previous sections the symbol m denoted the velocity ratio but in the following, we shall use it as $m = \frac{(\rho u^2)_\infty}{(\rho u^2)_0}$), and the compressibility parameter ω on the laws governing the evolution of the jet are established. Finally, the possibility of applying the method of the equivalent problem (in its simplest form) to such a flow was verified.

Analogous data in a more detailed form were obtained in a paper by V.Ye. Karelin [69,115]. Before we turn to these data, we must mention that it followed from L.P. Yarin's experiments that, with a given value of the parameter m, the decline curves of the quantities $\rho u^2 - (\rho u^2)_\infty$ and $\rho u c_p \Delta T$ on the jet axis are virtually coincident for all hot jets. In contrast to this, analogous curves for a practically isothermal jet differ from the others. This also indicates that the expressions for the transition from the physical coordinate to the reduced one in the case of hot jets were of the form $\xi = \xi(x, m)$ for values of $\omega > 1$ and with $\omega \gtrsim 1$ (and the same values of m) the relation between ξ and x was different from the others.

It was suggested to explain this difference by differences in the level of initial turbulence in cold and hot jets. In favor of the latter speaks especially the fact that in hot jets, regardless of the dependence on their degree of heating (with given value of m), the ρu^2 curves on the jet axis were virtually coincident. Of course, the heating of the gas was assumed to result in a higher level of turbulence compared to the cold jets. As will be shown below, V.Ye. Karelin's experiments verified this [69,115].

Figure 15.21 shows the experimental results on the decrease of momentum flux density and surplus heat content for three values of the parameter ω and $m = 0$ (efflux into quiescent air). For the same conditions, the corresponding functions $\xi(x)$ and $\xi_T(x)$ are shown. Just as in the other figures we had again a ratio of $\xi/\xi_T \cong 0.8$.

Analogous data for parallel jets (the curves showing $\rho u^2 - (\rho u^2)_\infty$ and $\rho c_p u(T - T_\infty)$ as functions of x/d_0 for one value of the parameter ω) have been plotted in Fig. 15.22. The data given characterize the influence of the two parameters of the problem, ω and m.

The next two figures, 15.23 and 15.24, are devoted to a comparison

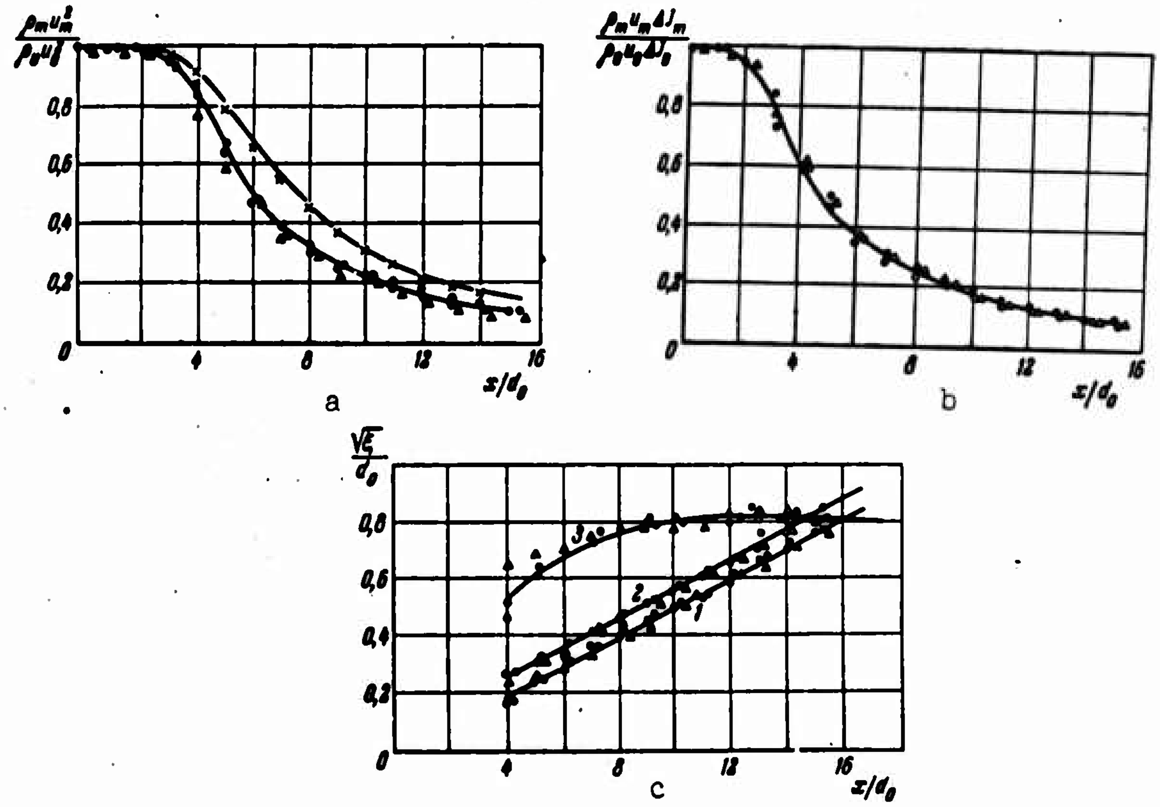

Fig. 15.21. Variation of momentum flux density (a) and surplus heat content (b) along the axis of a submerged nonisothermal jet. (Experiments by L.P. Yarin) × — T_0 = 300°K; ● — T_0 = 840°K; ○ — T_0 = 1240°K; △ — T_0 = 1520°K.

(c) 1, 2 and 3 — calculated curves of $\sqrt{\bar{\xi}}$, $\sqrt{\bar{\xi}_T}$ and ξ/ξ_T as functions of $\bar{x}$.

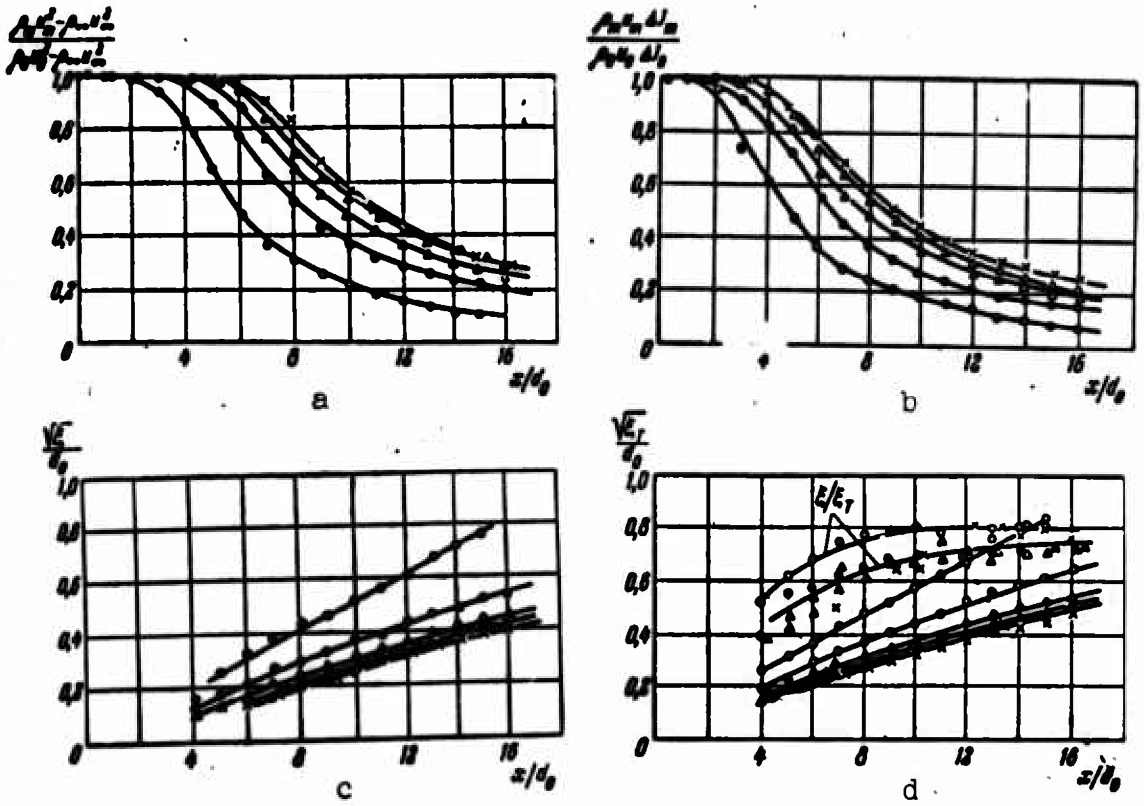

Fig. 15.22. Variations of surplus momentum (a) and heat content (b) along the axis of parallel nonisothermal flow with uniform distributions in the outlet section. (Experiments by L.P. Yarir). ○ — $T_0 = 1240°$ K, $m = 0$; ● — $T_0 = 1240°$ K; $m = 0.0445$; ▲ — $T_0 = 1230°$ K, $m = 0.150$; ▲ — $T_0 = 1240°$ K, $m = 0.280$; × — $T_0 = 1240°$ K, $m = 0.55$, where $m = (\rho w^2)_{\infty}/(\rho w^2)_0$; (c) and (d) calculated curves of $\sqrt{\varepsilon}$ and $\sqrt{\varepsilon_T}$ as functions of $\bar{x}$.

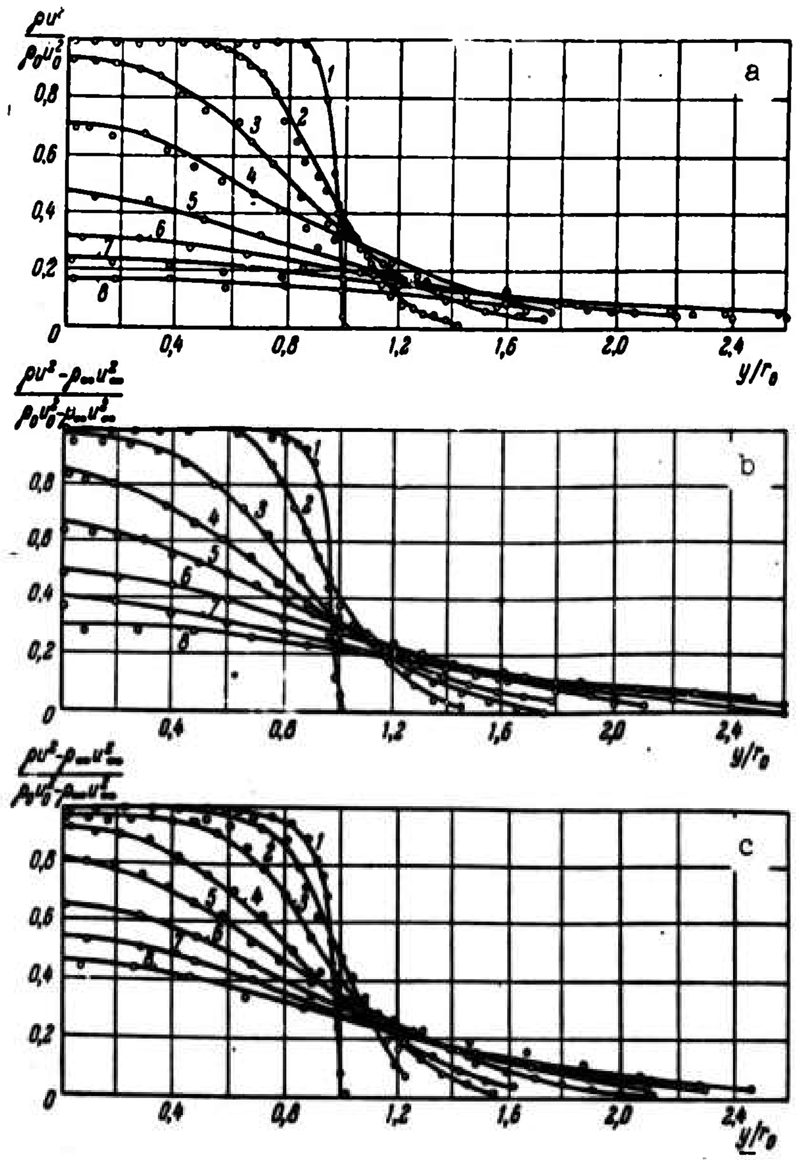

Fig. 15.23. Variations of surplus momentum in the cross sections of isothermal submerged and parallel jets. (Experiments by L.P. Yarin).
a) $m_{\rho u^2} = 0$; б) $m = 0,04$; в) $m = 0,16$.

№	1	2	3	4	5	6	7	8
x/d_0	0,05	2,0	4,0	6,0	8,0	10,0	12,0	14,0

ooo experiment; —— calculation according to the method of the equivalent problem.

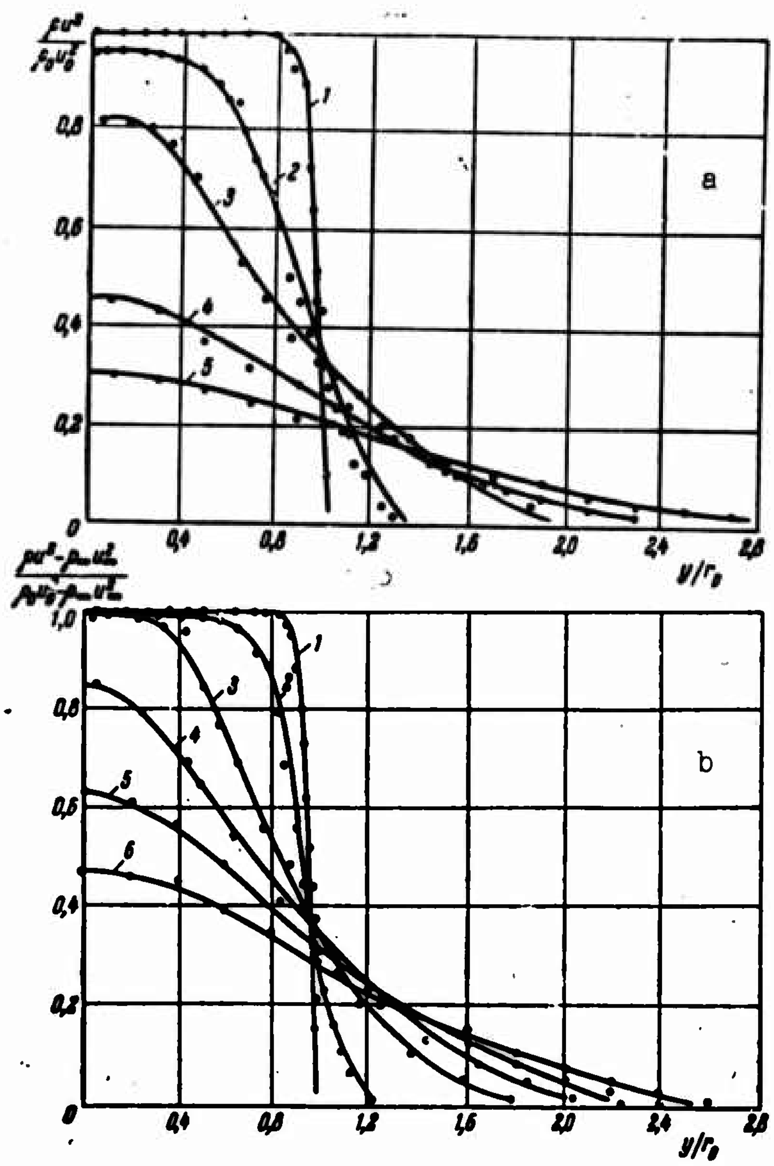

Fig. 15.24. Transverse distributions of ρu^2 in submerged nonisothermal jet (T_0 = 1280°K); b) transverse distributions of surplus momentum flux density in the cross sections of the parallel nonisothermal jet.

(T_0 = 1230°K; $m_{\rho u^2}$ = 0,135)

№	1	2	3	4	5	6
x/d_0	0,02	2	4	6	8	10

(oooo experiments by L.P. Yarin; —— calculation according to the method of the equivalent problem).

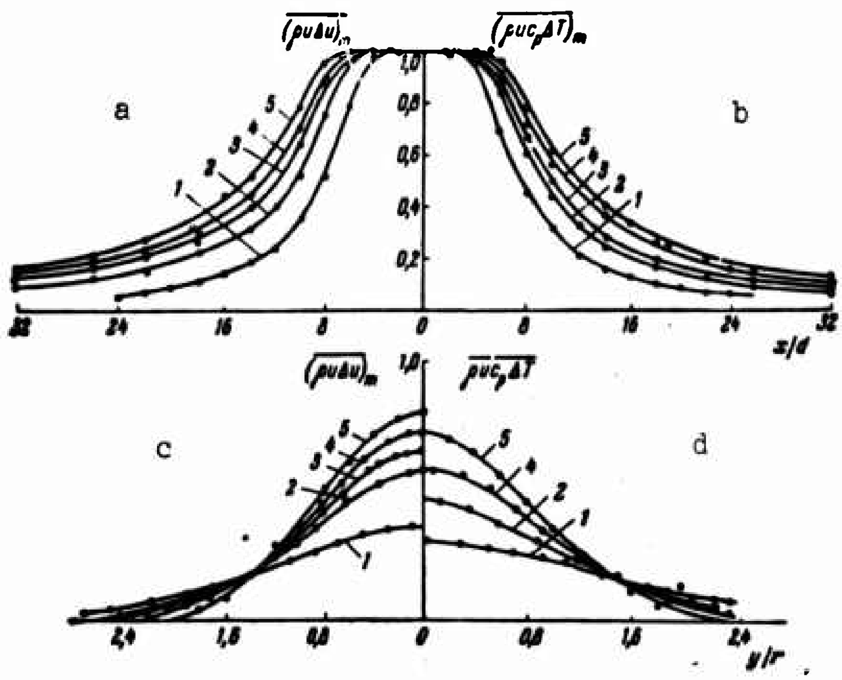

Fig. 15.25. Distributions of surplus momentum flux densities $\overline{\rho u \Delta u} = \rho u(u-u_\infty)/\rho_0 u_0(u_0-u_\infty)$ and flux densities of heat content $\overline{\rho u c_p \Delta T} = \rho u c_p (T-T_\infty):\rho_0 u_0 c_p (T_0-T_\infty)$ along the jet axis (a and b) and in the cross section (c and d). Experiments by V.Ye. Karelin [69, 115], $\omega = 1.2$.

№	1	2	3	4	5
m	0	0.04	0.10	0.17	0.2[illegible]

ooo experiments; —— calculation according to the method of the equivalent problem.

of experimental data (dots) and calculations according to the method of the equivalent problem (solid lines) for the distributions of the momentum flux density differences between jet and surrounding flow and the flux density of surplus enthalpy. Without entering into details which are clearly illustrated in the figures and the texts to them, we see that in all cases investigated, the agreement between experiment and calculation is fully satisfactory. In connection with this, the following may be remarked.

It is essential that when ξ is a function of x, the transverse

distributions of $\Delta(\rho u^2)$ (and $\rho u c_p \Delta T$) both calculated and measured, agree with a sufficient degree of accuracy. Under this condition, the relation between ξ and x implies the parameter m. In other words, in agreement with the physical content of the problem, the ratio between the quantities $(\rho u^2)_\infty$ in the flow and $(\rho u^2)_0$ in the jet enter the formula for the transition from the real space to the effective space. Precisely this makes it possible, when returning from the solution of the heatconduction-type equation to the real flow, to take the peculiarities of the latter into account: the curvature of the conventional boundaries of the jet, the drop in attenuation intensity of the jet as the value of $(\rho u^2)_0$ in the jet approaches the value of $(\rho u^2)_\infty$ in the flow, etc.

Let us now turn to the results of the paper by V.Ye. Karelin [69, 115]. In these studies (also under the conditions of a flame heating of the central jet) the initial temperature was allowed to vary between 300° and 1200°K, the velocity in the jet was up to 100 m/sec, the value of $m = \frac{(\rho u^2)_\infty}{(\rho u^2)_0}$ was between 1 and 4.2. In order to eliminate the influence of initial turbulence on the ρu^2 distributions and to obtain a single universal distribution in natural coordinates (i.e., without the introduction of any empirical constant), in the paper mentioned use was made of an additional device turbulizing the gas constituting the main jet. This additional turbulizer was made in the form of a tubular honeycomb (of tubes of a diameter of about 15 mm). In each tube a small vortex generator was arranged such that neighboring vortex generators produced whirls of opposite directions. In order to estimate the effect of action of the turbulence generator, we point out that with weakly heated air ($\omega \gtrsim 1.2$) thermoanemometrical measurements yielded a value of the order of 9% for the inten sity of velocity pulsations, compared to a value of about 2.5% without turbulence generator. In the

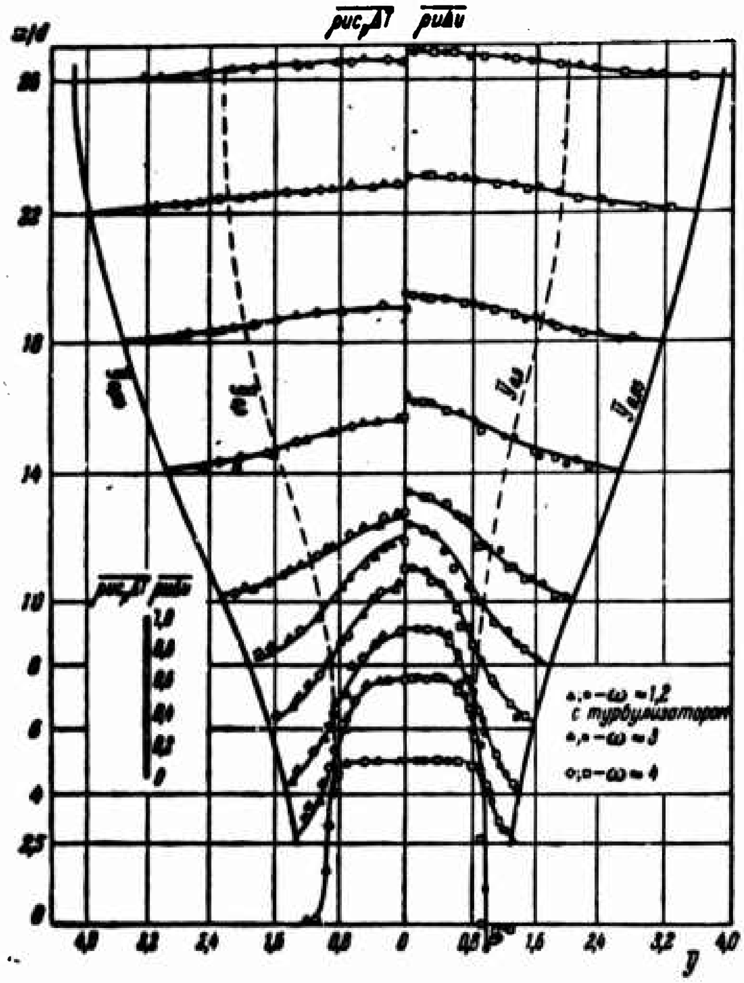

Fig. 15.26. Distributions of the values of $\overline{\rho u \Delta u}$ and $\overline{\rho u c_p \Delta T}$ in a parallel jet ($m = 0.17$, $\omega = 1.2$ [illegible]). ooo experiments; —— calculation according to the method of the equivalent problem. A) With turbulence generator.

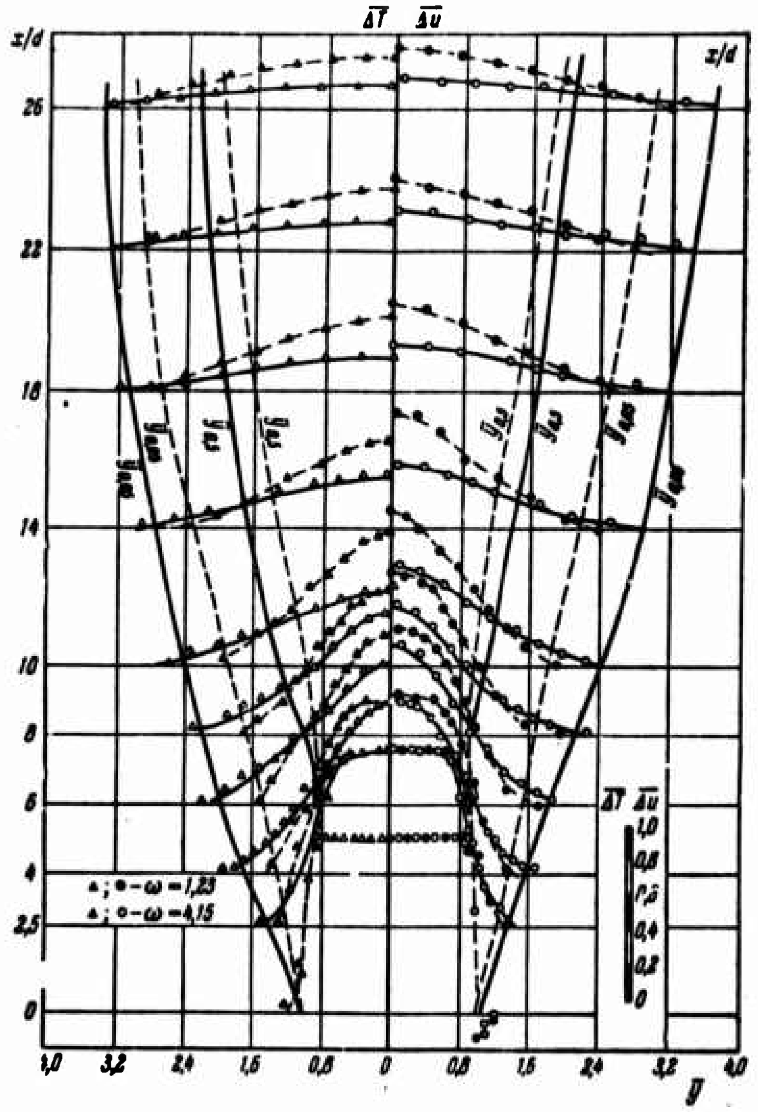

Fig. 15.27. Distributions of the values of velocity $\overline{\Delta u} = \frac{u - u_\infty}{u_m - u_\infty}$ and temperature $\overline{\Delta T} = \frac{T - T_\infty}{T_m - T_\infty}$ in a parallel jet — — — $m = 0.17$, $\omega = 1.23$; ——— $m = 0.17$, $\omega = 4.2$ [illegible]. (Lines: calculations according to the method of the equivalent problem.)

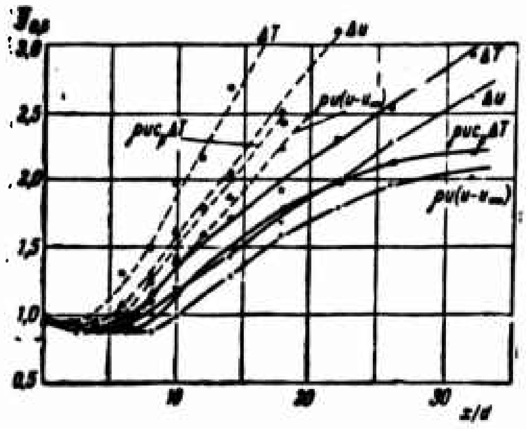

Fig. 15.28. Conventional "half-width" as a function of the distance determined for various parameters [69,115].
——— $m = 0{,}17$, $\omega = 1{,}15$; – – – $m = 0$, $\omega = 1{,}1$.

presence of a turbulence generator all the curves of $\rho u^2 - (\rho u^2)_\infty$, $\rho u(u - u_\infty)$, including those of the cold jet, for each value of m and different values of ω fuse to form a single universal curve. In the absence of a turbulizer, just as with the data given above, the isothermal curves differ essentially from the curves for the heated gas. To all appearance, the assumption on the direct influence of the pulsation intensity on the law of decline of ρu^2 in the jet is confirmed thereby. On the other hand, it was shown independently that with a sufficiently high level of turbulence, the parameter ω has virtually no influence on the distributions of momentum flux density and surplus heat content in gas jets (in any case within the limits of measuring accuracy). In particular, under conditions where all curves coincide i.e., with elevated initial turbulence, the distributions of $\rho u(u - u_\infty)$ and $\rho u c_p(T - T_\infty)$ are similar in the self-similar sections of jet flows but, consequently, there is no similarity between velocity and surplus temperature separately in jets with values of ω which differ essentially

from unity. More than that, for such self-similar sections, the value of the empirical constant a will be one and the same. We shall illustrate all this with the help of experimental data (first for flows with turbulence generator).

By way of example, we show in Fig. 15.25 the distributions of the relative quantities $\rho u(u-u_\infty)$ and $\rho u c_p(T-T_\infty)$ on the jet axis and in cross sections for different values of m and one and the same value of ω. The solid lines have been drawn after the results of solutions of heatconduction-type equations. Figures 15.26 and 15.27 show the distribution of the same quantities (dots: experiments, curves: calculations according to the method of the equivalent problem), the surplus velocity and the temperature. Their comparison permits the establishment of the laws of variation of these quantities, the conventional jet boundaries determined according to various features, etc. Note that the latter problem is of great interest. In connection with this, we show in Fig. 15.28 for two values of m and about the same values of ω the dependence of the conventional "half-width" of the jet (i.e., the ordinates of the points at which the value of, e.g., $\rho u(u-u_\infty)$ is equal to half the maximum value of this quantity in the cross section given) on the distance. The graphs given show clearly to which a high degree the values of $y_{\frac{1}{2}}$ determined in different ways differ from one another and in which a complex and uninterpretable way they depend on the parameters of the problem, particularly in the first part of the jet. As regards the total conventional width of the jet (e.g., the ordinate $y_{1/100}$ etc.), we can see from Figs. 15.26 and 15.27 that its determination from any of the characteristics would be very inaccurate. This can be explained by differences in the problem on the influence of the parameter ω on the jet boundaries and its treatment by various authors. The last two figures, 15.29 and 15.30, are devoted to the comparison of

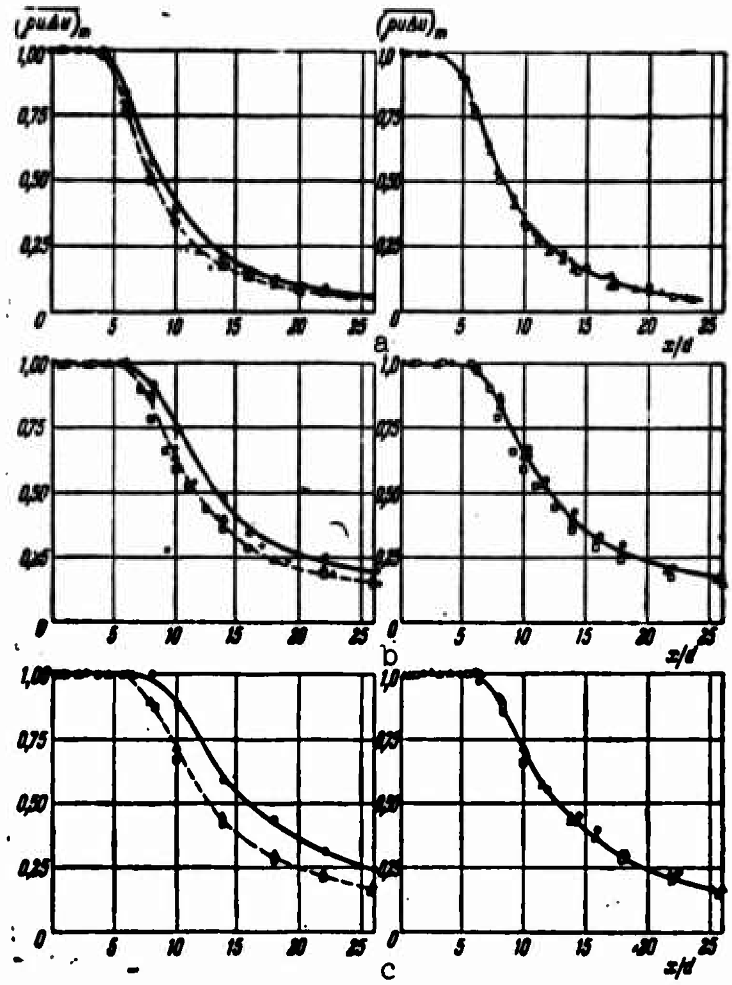

Fig. 15.29. Distributions of the value $\overline{\rho u \Delta u}$ on the jet axis. Left: without turbulence generator, right: with turbulence generator.
(○○); 2 (+ +); 3 (△ △); 4 (□ □) [,]: m = 0 (a); 0,1 (б); 0,17 (в); ● — 1,2
dots: experiments; —— calculation according to the method of the equivalent problem.

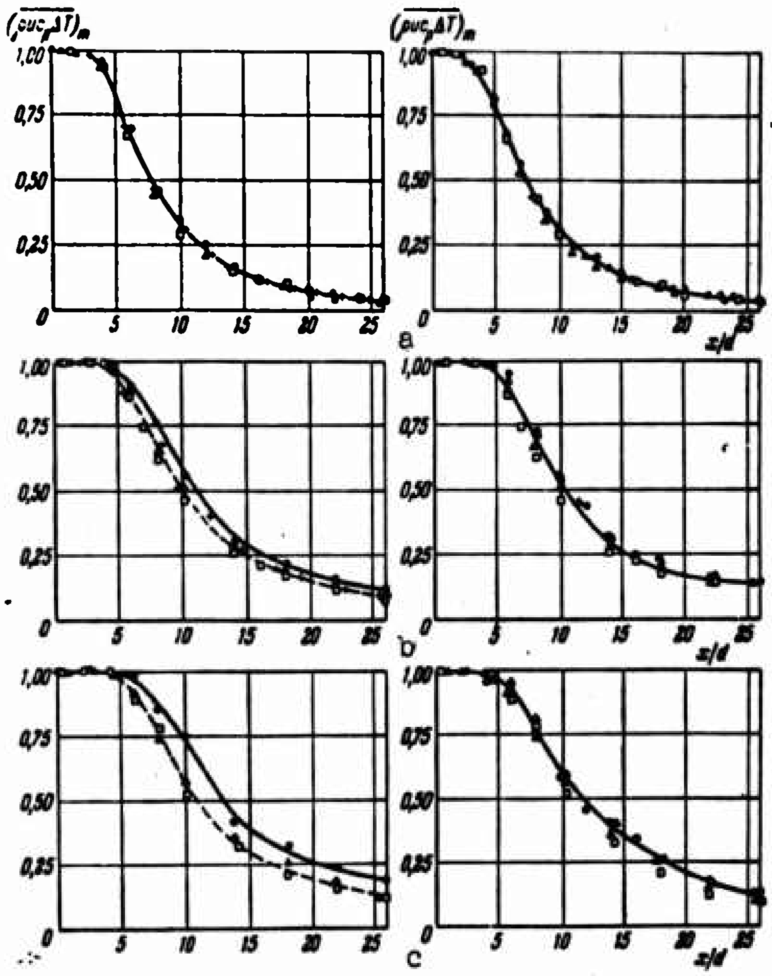

Fig. 15.30. Distributions of the value $\rho u c_p \Delta T$ on the jet axis [115]. (Denotations see Fig. 15.29). Dots: experiments;——— calculations according to the method of the equivalent problem.

experimental data obtained under the same conditions, but with or without a turbulence generator inserted in the jet. This has been discussed above. The solid lines in these figures represent the results of calculations according to the method of the equivalent problem. Thus, even with evolved turbulence and a universal distribution of $\rho u(u - u_\infty)$ (right) and with separation of the isothermal curves (left in Figs. 15.29 and 15.30) the calculation, with an appropriate

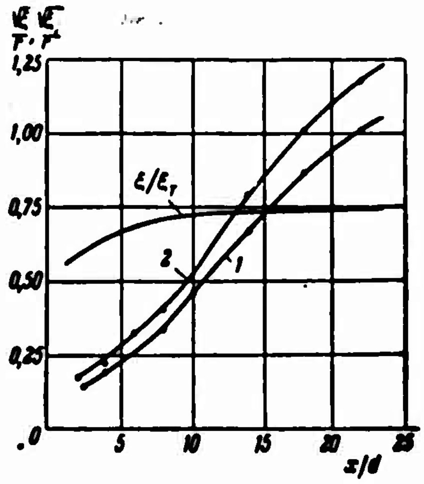

Fig. 15.31. Dependence of ξ (1) and ξ_T (2) on the coordinate x for parallel flow ($m = 0{,}17$, $\omega = 4{,}2$ [118]).

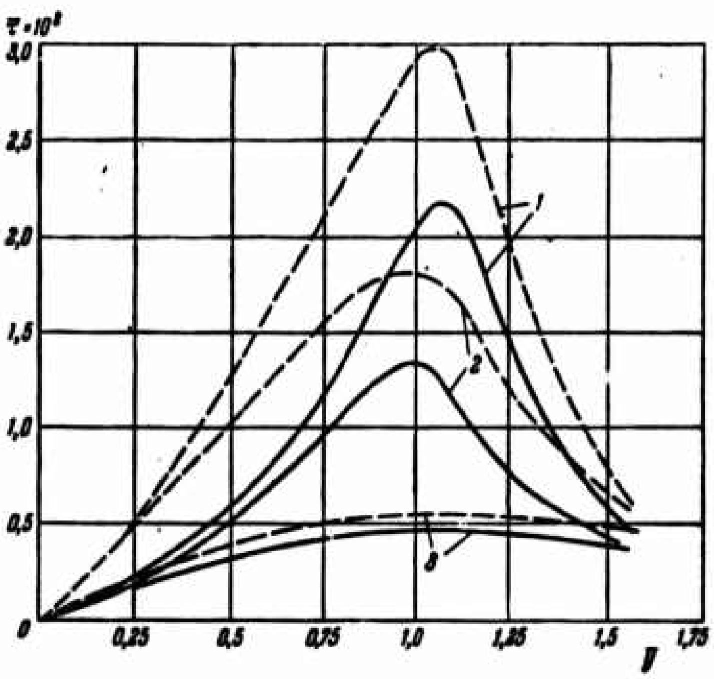

Fig. 15.32. Distributions of turbulent stress of friction $\frac{\tau}{(\rho u^2)_0}$ in the cross sections of an axisymmetric jet calculated according to the method of the equivalent problem

($m = 0$ {– – – $\omega = 1$; ——— $\omega = 4{,}1$ [119]). 1. $\frac{x}{d} = 4$; 2. $\frac{x}{d} = 6$; 3. $\frac{x}{d} = 12$.

function $\xi = \xi(x)$ — (taken from experiment), describes correctly the deformation of the distributions of momentum flux density and heat content in the entire range of variation of the parameters ω and m. By way of example, we show in Fig. 15.31 ξ and ξ_T as functions of x for the parallel flow shown in Fig. 15.26. According to their shape and the ratio ξ/ξ_T these curves do not differ from the many curves shown previously.

The successful application of the method of the equivalent problem to data on parallel gas flows yields a basis for their use and calculations according to the formulas given in Section 14.4, for the distribution of tangential frictional stresses. The results of this calculation for a jet issuing into quiescent air are shown in Fig. 15.32 for two values of the parameter ω (isothermal and limiting, for these experiments, heated jet). The curves of

$$\bar{\tau} = \frac{\tau}{(\rho u^2)_0}$$

represented in this figure for three values of the nozzle distance that, in our opinion, there is a real possibility of calculating the stress of friction, and of a subsequent systematic processing of the data according to the various jet flows, on the basis of the method of the equivalent problem. This is, of course, an empirical calculation. A similar but much less accurate result is obtained by a simple "inversion" of the boundary layer equations, i.e., they are used in order to calculate the frictional stress distribution from the given distribution (taken from experiment) of ρu^2, etc. But the considerably simpler determination of the quantity of friction (and also of the heat flux) on the basis of detailed experimental data, with the help of the two-term formula of the method of the equivalent problem permits the development of more complete representations of the transfer laws in turbu-

lent gas jets. This problem belongs to the field of further investigations.

15.6. SOME REMARKS

Let us consider some results.

First of all, it should be noted that the various experimental data given and their processing by means of the method of the equivalent problem verify the expediency of the application of this method to the calculation of turbulent, plane, axisymmetric and spatial jets, submerged and parallel, with constant and variable densities, arbitrary form of the initial distributions, etc. The experimental material also illustrates the foundation of the method discussed previously and its difference from the previous modifications and, first of all, from H. Reichardt's method. The weak point of the latter, besides the unfoundedness of the basic equation, is the linearization of the boundary layer equations (already mentioned in literature, see, e.g., the book by I. Hintze [200]), the identification of submerged and parallel flows (for the surplus quantities) and the like in the physical plane of flow. This results in a direct contradiction with experiment and an ignoring of the actual peculiarities of the various problem.

In contrast to this, the method of the equivalent problem keeps to the real nonlinear flow with all its peculiarities and takes them into account when choosing from experiment the mathematical "key," the transition function to the effective space in which the flow, in a well-known approximation, may be described by a linear equation of the type of the heatconduction equation. In this case, one and the same solution to this equation corresponds to a great number of real flows in a physical plane, which is connected with the effective plane by appropriate equations, in the simplest case by the single expression $\xi = \xi(x)$. An arbitrary peculiarity of the real problem, the

form of the initial distribution, here including the properties of the parallel flow, the "history" of the flow, i.e., the form of the nozzle, the presence of a turbulence generator, etc., all this has finally an influence on the experimental relation linking ξ and x. In particular, in gas jets, when returning from the effective plane to the physical one, we also have a correlation between the formally independent equations for the dynamic and the thermal problem.

Let us also mention that the presence of a set of curves $\xi = \xi(x)$, and $\xi_T = \xi_T(x)$ for different flows, enables us – and such attempts have already been made in the papers cited – to use the method of the equivalent problem in approximate interpolation calculations for recalculations, i.e., for the development by way of calculation, without a direct experiment, an approximate flow pattern for noninvestigated jet motions. This possibility will increase undoubtedly when more and more experimental data are collected. But this must not give the illusion that this method is "universal," especially in its simplest form. In a series of cases of complex flows, semilimited jets, antiparallel jets, the appearance of dynamic effects resulting in a fusion of parallel jets, their "adhesion" to surfaces of solids, and the like, the simple variant of the method (invariant coordinate y) may only yield a qualitative flow pattern. To obtain the quantitative pattern, it is necessary to expand the volume of empirical information. The expediency of a further expansion of the field of applicability of the method will then depend to a high degree on whether we can find rational forms for it. In certain cases, it will be effective to pass over to the coordinates $\xi = \xi(x)$, $\eta = \psi$, where ψ is the streamfunction.

It is probable that in various concrete cases, even the ideas of a rational form of calculation will diverge, i.e., the relations between the data taken from experiment and those from the theoretical

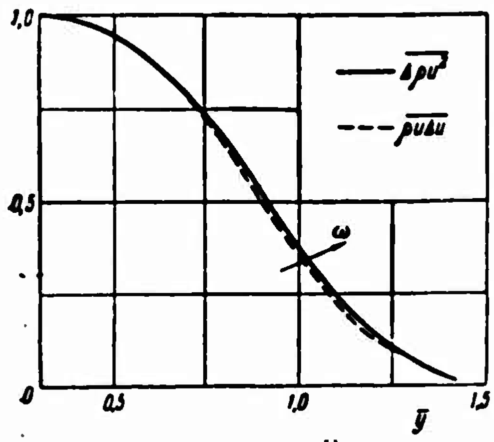

Fig. 15.33. Comparison of the distributions of $\overline{\rho u\ \Delta u}$ and $\overline{\Delta\rho u^2}$ in the jet cross section ($m = 0.1$; $\omega = 2$; $\frac{x}{d} = 6$ [118]).

solutions will differ.

As shown above, in all cases studies the value of the ratio ξ/ξ_T proved to be smaller than unity, of an order of about 0.8. By the way, this means that the calculation according to the method of the equivalent problem, in agreement with experience, results in the fact that heat transfer occurs more rapidly than momentum transfer. This indicates that in integral-adiabatic gas jets, there exists a local redistribution of total enthalpy. The character of this distribution corresponds qualitatively to the case where the Prandtl number is smaller than unity: the faster striae becomes enriched in energy at the expense of the slower ones. Examples of such a distribution were given previously.

It should be borne in mind that in the previous sections we used two ways of processing the experimental data on the momentum flux density for a jet expanding in a parallel stream. In the analysis of V.Ye. Karelin's experiments (Figs. 15.25-15.32) the quantity $\rho u(u - u_\infty)$.

was chosen as the basic characteristic, a quantity which is encountered everywhere. This choice is quite correct as it is precisely this quantity which enters the integral condition of conservation of the form

$$\int \rho u (u - u_{\infty}) ds = \text{const},$$

which results immediately from the heatconduction-type equation (by way of integrating it across the jet). In earlier experiments (I.B. Palatnik et al.) instead of $\rho u \Delta u$ the quantity $\Delta \rho u^2 = \rho u^2 - (\rho u^2)_{\infty}$. was introduced in the analysis. Though this is not strict (as no conservation condition can be written for $\Delta \rho u^2$ it is convenient for practice as the determination of $\Delta \rho u^2$ in experiments is not connected with any additional measurements, except for the quantity ρu^2.

It is essential that within the limits verified experimantally the difference between the nondimensional distributions of the quantities $\rho u(u - u_{\infty})$ and $\rho u^2 - (\rho u^2)_{\infty}$ are insignificant. By way of example, we compared in Fig. 15.33, the urves of both characteristics, $\overline{\rho u \Delta u}$ and $\overline{\Delta \rho u^2}$. for a nonisothermal jet. The comparison shows that the difference between them (especially in heated jets) is within the experimental limits of accuracy. Thus, though we consider the processing with respect to $\rho u \Delta u$ more correct in principle and sometimes, for example, in a hot burning jet expanding in a parallel flow (see Chapter 17) it is the only possible way, we must not give up the approximation method convenient in practice.

The experimental data given refer to subsonic gas jets. We see that the method of the equivalent problem also applies to calculations of the mixing zone in high speed jets (comprising supersonic ones). This is verified by individual experimental data on gas jets given in the literature referred to and also in a recent paper [33] dealing

with jets of hot steam. A systematic verification and generalization of experimental data on supersonic gas jets within the framework of this plan has so far not been carried out.

It should also be noted that in the papers mentioned, which deal with the application of the method of the equivalent problem, the method of calculation according to a heatconduction-type equation has been improved. In the first papers [34, 149] the main mathematical aids were often only integrators of various types. In spite of their clearness, the increased experimental accuracy rendered it necessary also to improve the mathematical procedure. An aid well known in practice for the development of solutions to the axisymmetric problem may be the tables [264] mentioned above.

In particular, when we introduce in the calculation the actual initial distribution of ρu^2 (which, in the general case, is nonuniform) etc., even the accounting for relatively small deviations, e.g., from a rectangular distribution, improves the agreement between theoretical results and experiment (especially near the nozzle) as shown by V.Ye. Karelin [69, 115].

Summing up we may say that the method of the equivalent problem, though it is of course not universal, undoubtedly has well known advantages compared to other semiempirical methods of calculation.

Of course, this does not mean that not other methods of calculation must be used in a series of cases in which it is unnecessary to know the detailed velocity and temperature distributions (and their continuous deformation under the conditions of nonself-similar jets). In connection with this, besides the widespread mathematical methods developed by G.N. Abramovich [11] it is worth mentioning the method of the layer of constant thickness applied in the theory of turbulent jets. This way of calculation of turbulent jets is connected with the

use of integral relations derived successively in the papers by A.S. Ginevskiy [75-79]. In the past years A.S. Ginevskiy applied the method of the integral relations to the solution of a great number of problems on the expansion of free, turbulent, isothermal and nonisothermal jets of fluid and gas.

Using the well known integral relations by V.V. Golubev, A.S. Leybenzon et al. and approximating the velocity and temperature distributions (or the distributions of frictional stress and heat flux, see Section 4.2 for laminar jets) it is possible in a series of cases, by means of the introduction of the corresponding empirical constants, to achieve satisfactory agreement with the experiment. A weak point of this method, apart from that indicated in Chapter 4, is the fact that Prandtl's formula for the tangential frictional stress $\tau = \rho l^2 \left(\frac{\partial u}{\partial y}\right)^2$ is applied in the case of a compressible gas jet. At the same time, the method of the integral relations, just as also the calculations based on a direct assignment of the distributions of velocity, etc., is valuable in practice as it makes it possible somewhat to increase the number of problems of the theory of turbulent jets, which are accessible to calculation.

REFERENCES

4, 8, 11, 21, 22, 25, 28, 31, 33, 34, 35, 38, 39, 40, 41, 58, 61, 64, 67, 69, 72, 80, 82, 83, 88, 96, 98, 111, 114, 115, 135, 136, 137, 138, 149, 150, 154, 157, 178, 186, 187, 190, 193, 194, 195, 199, 200, 210, 211, 214, 217, 218, 264, 266, 269, 270, 281, 321, 322.

Manuscript Page No.	[Footnotes]
352	Analogous data are contained in the papers [183]. See also Section 15.5.
358	As already mentioned, from the physical point of view, it would have been more correct to use the quantity $\rho u (u - u_\infty)$ as the "surplus momentum" for a jet in a parallel flow. For the problem considered this yields virtually the same results as obtained by more cumbersome calculations.

Part Four

SOME SPECIAL PROBLEMS

The last part of this book has been devoted to the discussion of some essential problems of the theory of jets of viscous fluid and gas and their application. These problems are closely related to those considered in the preceding sections but which, at present, are less far developed.

We shall discuss in particular (pre-eminently in the experimental plan) individual complex jet flow which are very important in practice, namely the antiparallel flows and jets streaming around bodies. The method of solving such problems theoretically have not yet been developed sufficiently. The chapter devoted to them will therefore mainly contain experimental data and their generalizations.

A separate chapter has been devoted to the diffusion heating of gas. The method of the theory of jets is here applied to the calculation of laminar and turbulent combustion of previously nonmixed gases. It can be shown that an application of the method of the equivalent problem of the theory of thermal conductivity developed in the preceding part of the book increases essentially the possibilities of calculation in one of the fundamental fields of the gas dynamics of combustion, namely the theory of the turbulent burning jet.

In the last chapter we made an attempt of classifying the solutions of a series of problems on the expansion of jets of electrically conducting fluids which are interacting with a magnetic field. As there is absolutely no experimental material available, the solution of these problems has mainly methodical significance as this is shown by the specific nature of the magnetogasdynamic consideration on the one hand and its connection with the general theory of jets, on the other.

Chapter 16

SOME COMPLEX JET FLOWS

16.1. PRELIMINARY REMARKS

In the preceding parts of the book, we considered the simplest and idealized cases of jet flows of viscous fluids and gases. This type of problems, dealing with the expansion of source jets, plane. fan-type and axisymmetric, free and semilimited jets and the like, are fundamental for the investigation of real jet flows. The circle of problems which can be analyzed theoretically from a single point of view is enlarged considerably when we take into account the finite dimensions of free jets, the continuous deformation of arbitrary initial distributions, the expansion of the jet in a parallel flow, in short, all what is connected with the method of the equivalent pro blem. But also here most of the quite different jet flow problems important in practice are without the framework of investigations.

In the various fields of engineering one encounters peculiar problems of jet flow investigations which are mostly connected with such complex and hardly studied questions as, for example:

1) the laws of expansion of a jet in a flow whose direction with respect to the flow is arbitrary;

2) the collision and intersection of jets;

3) the expansion of a jet in a limited space;

4) the expansion of strongly twisted free or limited jets (for example, in a cyclone chamber [114, 193] etc.);

5) the interaction of jets with solids, in particular, the jet flow around bodies, the effect a jet exerts on an obstacle and many others.

An investigation of these problems is often of immediate interest in practice. This explains the interest which complex jet flows meet in the literature on applied gas dynamics or in monographs and articles devoted to concrete technical installations.

As a rule, we are concerned with the experimental investigation of the problem, the gathering of data in the one or other measuring range of parameters specific of the system's conditions and others. As regards the generalization of the experimental material and the classification of the results obtained, this is, to an essential degree, a future task. We therefore refer the readers to the numerous experimental data contained in, e.g., the papers by M.A. Glinkov [81], Yu.V. Ivanov [107-111], D.N. Lyakhovskiy [136-141] and others, and the detailed reference lists given in these papers (partly entered in the literature index at the end of this book) we restrict ourselves to a very brief consideration of a few examples of complex jet flows. As the latter, we choose individual acute problems which are also characteristic, taken from the circle of investigations carried out under the leadership of one of the authors. In this connection, we think of the problem of jet expansion in an antiparallel flow and the flow in the wake behind a body located in a uniform flow or jet.

These problems, as we shall see in the following, are closely interrelated. Though their theoretical solutions (by means of the same semiempirical methods) are not available, a general representation of the qualitative flow structure can be obtained by means of a roughly approximative way of "constructing" the flow (by the method of superimposing jets, addition of dynamic pressures and other qualitative

procedures applying quite unaccurately to the motion of viscous fluids).

As regards the other cases of complex jet flows which are no less important in practice, they will not be considered here in agreement with the general layout of the book. We shall also not touch the problem of two-phase jets, the problem of jets carrying along solid or liquid particles [36, 37, 205, 206] or other special problems of the theory of turbulent jets and their various applications.

16.2. ANTIPARALLEL JETS

A detailed qualitative representation of the character of motion arising on the efflux of a jet against a uniform stream can be obtained if one has recourse to the superposition of motions (which is incorrect in the case of viscous fluid flows). In practice we have several ways of doing this, for example, by adding the stream functions of jet and flow or, what is essentially the same, by means of a geometrical addition of the velocities of the two conditionally independent flows. Finally, somewhat better (but also qualitative) results are obtained when the dynamic pressures are added.

These ways of a "constructive" buildup of complex flows and other similar modifications as a rule transfer correctly the fundamental specific peculiarities of a complex flow, for example, the formation of a recirculation zone in a nonpotential flow around a body or an opposite source jet. Recall in this connection that a series of flow conditions for the problem on the mixing of two antiparallel flows was obtained previously [51] when the problem was solved by the superposition method.

Thus, the question is the obtaining of a preliminary pattern of a complex flow. As regards the comparison with experimental data, for the

problem considered here (antiparallel flows) the introduction of a single empirical constant is only sufficient for the approximate description of the laws of flow along the jet axis. In order to achieve full agreement between the calculated flow pattern and the actual motion, it would be necessary to increase the number of empirical constants taken from experiment. The same is true in essential for the application of the method of the equivalent problem of the theory of thermal conductivity. A successful application of this method to antiparallel jets requires the transition to more complex formulas of transformation of variables of the form $\xi = \xi(x), \eta = \eta(y)$ (or even of the form $\xi = \xi(x), \eta = \eta(x, y)$). The development of an effective method of calculating antiparallel jets is therefore still incomplete. At the same time this problem is acute for the combustion and furnace techniques, etc., in particular, in connection with the problem of the jet stabilization of a flame in a high-speed flow. This fact also stimulated the investigation of antiparallel jets [114].

Figure 16.1 shows by way of example experimental data on the expansion of an axisymmetric air jet in an unlimited counterflow. Besides the velocity distributions in the flow cross section, the figure also shows the characteristic curves bounding the recirculation zone, the

Fig. 16.1. Velocity distributions in cross section of axisymmetric jet expanding in a counterflow. 1) m/sec; 2) m.

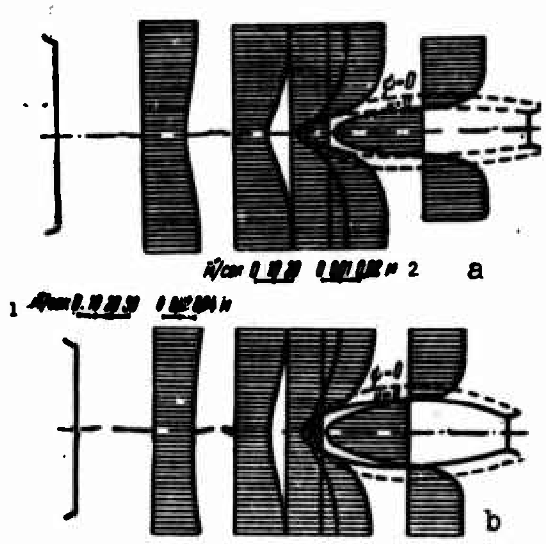

Fig. 16.2. Construction of flow pattern for a jet in a counterflow (after the data of Fig. 16.1). a) Addition of velocities; b) addition of dynamic pressures. 1) m/sec; 2) m.

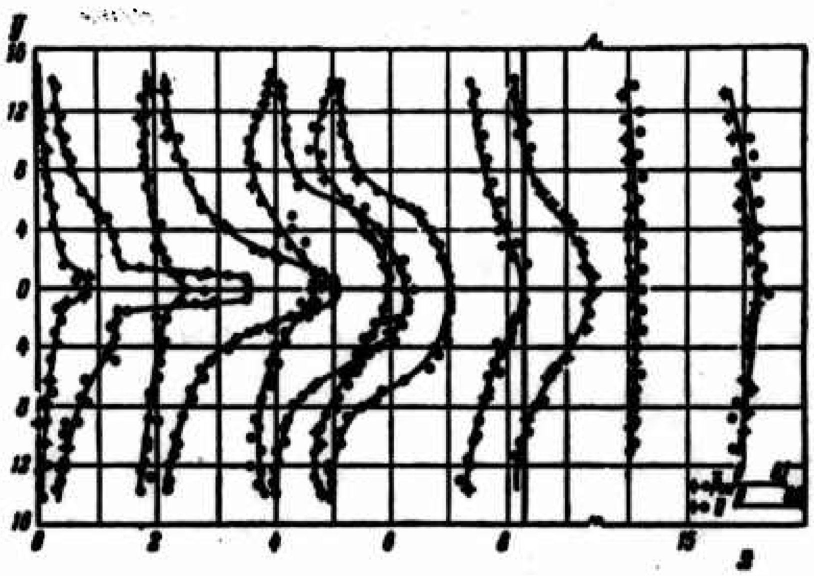

Fig. 16.3. Velocity and pressure distributions in a plane jet expanding in a counterflow with $m = -0.5$ (see [127]).

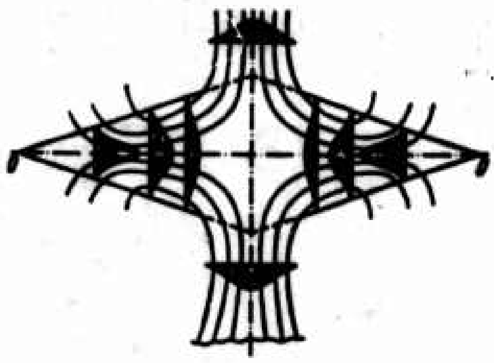

Fig. 16.4. Diagram of flow being the result of two antiparallel streams (or antiparallel fan-type jet).

lines of zero longitudinal component of velocity ($u = 0$) and zero streamline ($\psi = 0$).

For the purpose of comparison we show in Fig. 16.2 the flow patterns calculated for the experimental conditions of Fig. 16.1, constructed by means of an addition of velocities (Fig. 16.2, *a*) and of dynamic pressures ρu^2 (Fig. 16.2, *b*). From the qualitative point of view the agreement of the results is obvious.

We may see from partial results of the investigation that the relative length x_0/d of the counterflow zone is proportional to the ratio u_0/u_∞ where u_0 is the efflux velocity of the jet. The latter, however, follows from dimension considerations.

Owing to the curvilinear shape of the trajectories, the motion investigated (jet in counterflow, antiparallel jets) is connected with a noticeable variation of pressure and, in individual sections, with a commensurability of the longitudinal and transverse velocity components. Generally speaking, owing to these peculiarities, the methods of the boundary layer theory can only be applied under certain conditions to the problem of antiparallel jets. In contrast to this, a more

"quiescent" complex flow, as the expansion of a jet in a parallel flow, can be described satisfactorily, not only qualitatively but also quantitatively, by "superposition" methods, just as by the method of the equivalent problem.

The most careful experimental investigation of the motion connected with the expansion of a plane jet in a uniform counterflow was carried out by T.P. Leont'yeva and B.P. Ustimenko [127] from whose paper the above Fig. 16.3 was taken.

An analogous way of "superposition" can be used for the construction of the qualitative flow pattern produced by two antiparallel plane jets (of equal initial momenta), or for an axisymmetric flow produced by a convergent fan-type jet. A schematic representation of this pattern is shown in Fig. 16.4.

An investigation of such jets, their interactions (e.g., the action on a free jet of transverse striae, etc.) are of great interest in practice, in particular for gas jets.

16.3. FLOW PATTERN IN THE WAKE BEHIND A BODY

It is well known that the motion in the wake behind a body of a badly streamlined shape, far away from it, has a marked jet-type character and (in the case of laminar and turbulent flows) can be calculated by the method of small perturbations. We do not give here these solutions as they have been considered in detail in many monographs [11, 130, 174, 194, 212, etc.] but deal with the flow in the wake immediately behind a badly streamlined body.

A systematic pattern of such a flow was constructed by the method of "superposition" of the fictive turbulent momentum flow and the uniform flow [58]. As we see from Fig. 16.5, the flow is characterized by the zone of backflows, with the same limiting surfaces $\psi = 0$ and $u = 0$.

For the region of an evolved turbulent flow (with a value of the Reynolds number of the order of $Re \geqq 10^4 + 10^5$) a calculation by means of the above method permits an estimation of the length of the backflow zone in the uniform flow around bodies of various forms. The theoretical results agree qualitatively with the experiments and when a single empirical constant is introduced, the agreement is even quantitative.

For a plane flow, the reduced length of the backflow zone is equal to

$$\frac{ax_0}{d} = \frac{3}{8}\zeta,$$

where ζ is the drag coefficient $a \approx 0,35.$

For axisymmetric bodies

$$\frac{ax_0}{d} = \sqrt{\frac{\zeta}{8}}.$$

These formulas describe with qualitative correctness the influence of the geometrical shape of the body on the character of the flow around the body and, particularly, on the length of the backflow zone. For example, in the case of bodies with smooth surfaces placed in the flow (cylinder in transverse flow $\zeta \approx 1,0 + 1,2,$ and sphere ζ = 0.5) the zone length is equal to $\frac{x_0}{d} \approx 1,3$ and $\frac{x_0}{d} \approx 1,2.$ respectively.

The crisis in the flow around a cylinder or sphere is accompanied by a drop of the drag coefficient which, as resulting from the formulas, causes a corresponding reduction of the dimensions of the backflow zone.

In the case of bodies with sharp edges (plate in transverse flow, $\zeta \approx 2,0,$ and disc $\zeta \approx 1,1$) the relative zone length is equal to $\frac{x_0}{d} \approx 2$ in the first case and $\frac{x_0}{d} \approx 1,9$ in the second, i.e., it is considerably higher than with bodies of smooth surfaces.

A qualitative verification was also achieved for the conclusions on the self-similarity of the averaged flow, the independency of the

backflow zone dimensions on the velocity of the incoming flow etc., which results immediately from the calculations.

Note that at first these results which prove the correctness of the average turbulent motion in the wake behind the body were contradictory to the previously widespread idea of a supposedly total chaotic state of this motion.

An analogous scheme was used by S.I. Isatayev [112, 193] in order to construct the flow pattern for a jet flow around a body. It was shown in particular that the length of the backflow zone in plane and axisymmetric motions is proportional to the distance between the pole of the jet and the badly streamlined body [112].

In the limiting case where this distance is allowed to grow unlimitedly, the formulas obtained were reduced to the above expression for the uniform unlimited flow around bodies. More than that, this simple calculation permitted the prediction of another characteristic peculiarity of a jet flow around a badly streamlined body, namely the possible appearance of an "open" flow pattern for an infinitely long backflow zone. This effect is actually observed with sharp-edged bodies placed in a jet flow. In the case of bodies with smooth surfaces, in contrast to this, the possibility of a smooth and virtually potential flowing around them, of a turbulent jet, results from the calculation and was verified by experiment (see below).

The dependence of the length of the backflow zone on the distance between a badly streamlined body and the jet nozzle is shown in Fig. 16.6 for a plate and a cylinder.

The peculiarities of a jet flow around bodies are so unusual and significant in practice that they deserve a more detailed discussion on the basis of experimental results.

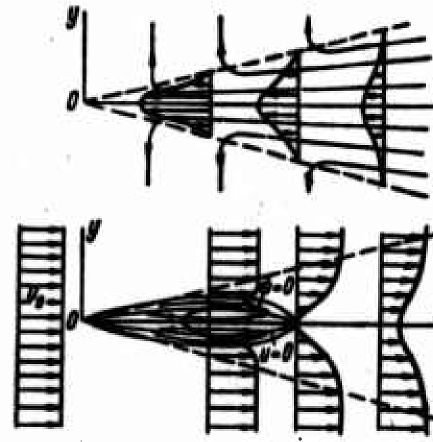

Fig. 16.5. Schematic representation of the formula of a circulation zone behind the body. a) Motion in "turbulent flow;" b) motion behind a body, simulated by point discharge of momentum.

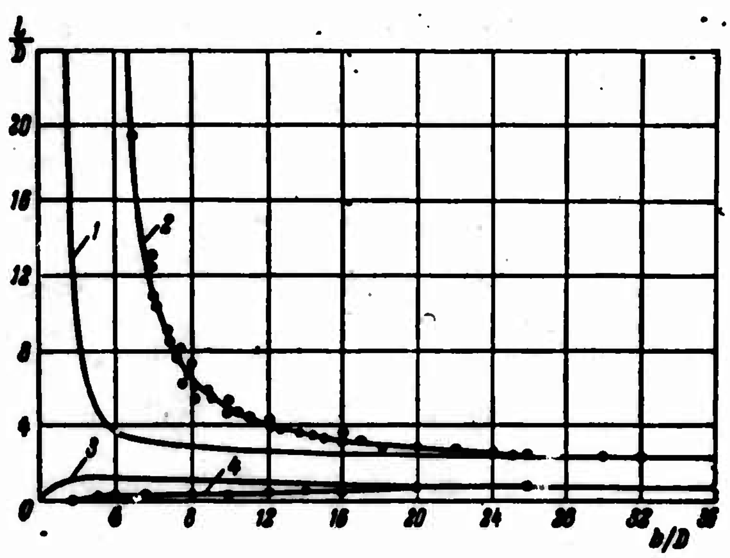

Fig. 16.6. Length of the backflow zone as a function of the distance between nozzle and body [112, 113].

1) calculation, 2) experiment, } a) for a plate; 3) calculation, 4) experiment, } b) for a circular cylinder.

16.4. JET FLOW AROUND A BODY

Let us report briefly on the most important results of a comprehensive experimental investigation carried out by S.I. Isatayev [112], which deals with the study of uniform flows and turbulent jets streaming around badly streamlined bodies.

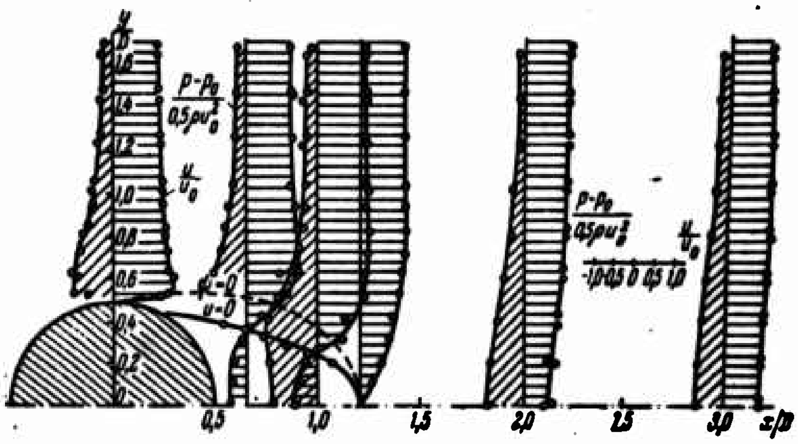

Fig. 16.7. Uniform flow streaming around a circular cylinder (data from [114]).

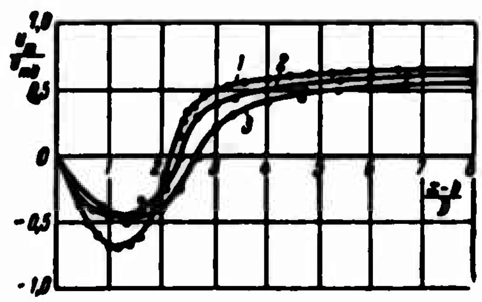

Fig. 16.8. Velocity distribution on the wake axis in uniform flow (1) or jets (2 and 3) streaming around a plate of dimension *D*, at a large distance, equal to *b*, from the orifice [112, 113]. 1) *D* = 50 mm; 2) *D* = 15 mm, *b/D* = 32; 3) *D* = 15 mm, *b/D* = 24.

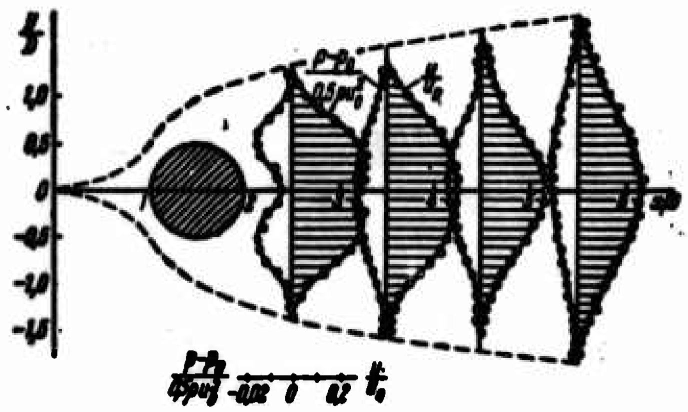

Fig. 16.9. Velocity and pressure distributions in the wake behind a cylinder placed in a jet flow *(D = 40 mm, u_0 = 57 m/sec* [112]).

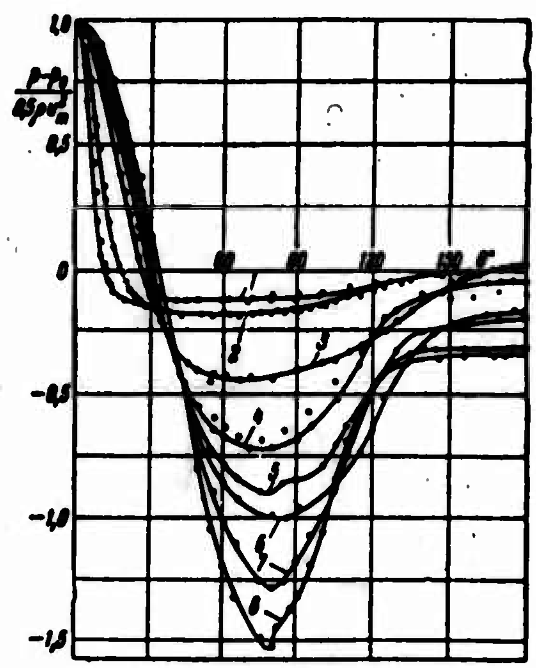

Fig. 16.10. Pressure distribution on the surface of a cylinder placed in a jet flow [112]; *D = 73 mm, u_0 = 23-57 m/sec.*

№	1	2	3	4	5–6	7	8
b/D	0.7	1.4	2.7	4.0	5.5	8.2	10.5

As shown in papers [59, 112, 114], in a jet flow around bodies with sharp edges, two characteristic types of flows may take place, closed and open ones, which may pass over to one another when the position of the body in the flow is changed. When the body approaches from a very great distance (flow around as in the case of a uniform flow) the jet nozzle (to a distance of 4-5 nozzle diameters for a plate and 3-4 for a disc) the dimensions of the backflow zone will increase smoothly at first and then impetuously until the flow has become closed. This is then maintained independently of a further approach.

When the body is displaced in the opposite direction, i.e., when it is removed from the nozzle orifice, the reformation of the open flow pattern to the closed one occurs with a considerable retardation, i.e., at a greater distance (of the order of 7-8 nozzle diameters for the plate and 5-6 for the disc) from the nozzle than when the closed flow pattern changes over to the open one. Thus, the change of the flow patterns occurs with a peculiar "hysteresis" (it is essential to know the history of the flow in order to ascertain the flow conditions). This is a curious effect which, as shown by experiment, is self-similar, that is, it is independent of the initial velocity of the jet. The flow in the region of the hysteresis is unstable and the type of flow around the body may be changed by the influence of relatively small perturbations.

In Figs. 16.7-16.11, in order to illustrate the conditions, we give some data characterizing the jet flow around badly streamlined bodies (Figs. 16.9-16.11) and, for comparison (Figs. 16.7-16.8), the uniform transverse flow around a circular cylinder. The experimental data corresponding to the curves in the figures are given in the text to the latter.

It must be remarked that, as to their physical nature, these ef-

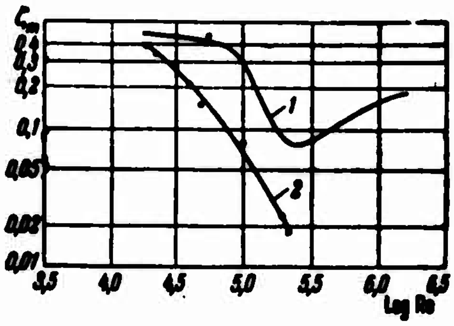

Fig. 16.11. Drag coefficients of a sphere as a function of the Reynolds number. 1) Uniform flow around it [152]; 2) jet flow around it [113].

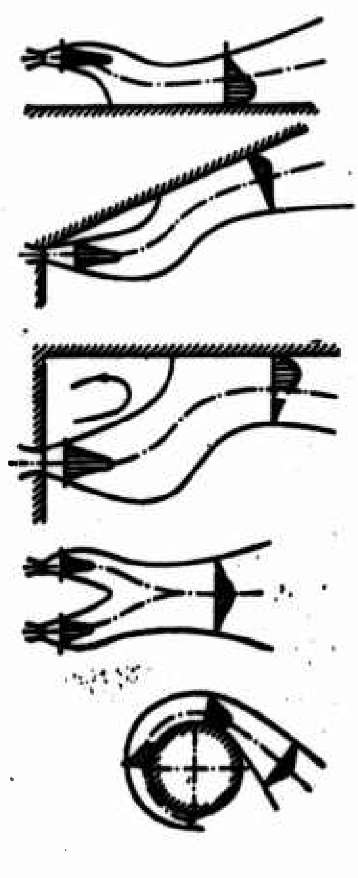

Fig. 16.12. Schematic diagram of "adhesion" of jet to solid wall.

fects, especially the potential flow around a badly streamlined body (cylinder, sphere) produced by a turbulent jet, resembles effects mentioned in literature. These are the "adhesion" effect of the jet to the front sheet of a nozzle which D.N. Lyakhovskiy denoted incorrectly* "kinematic ultradiffuser" [139] and other effects reported in American literature [232] as the Koand [sic] effect (named after the inventor). By way of example, we show the adhesion of a jet to a solid surface shown schematically in Fig. 16.12. In these cases and other analogous cases, the explanation of the effect observed is connected with the dynamic flow pattern, i.e., the pressure distribution. In the absence of symmetry, the inleakage of fluid from the "tapered" region is accompanied by a stronger rarefaction than from the unlimited space. The pressure drop thus arising may, although it is small in absolute magnitude, cause a curvature of the jet and its adhesion to the surface.

The same occurs, for example, in the case of the adherence of two parallel plane jets issuing at a certain distance from one another (this flow was mentioned previously). The possibilities of partially calculating such a flow according to the method of the equivalent problem were indicated above. They also refer to other analogous cases. We see from the examples given that the nature of the effect, strictly speaking, is without the framework of the boundary layer theory as it is connected with a violation of pressure constancy across the layer. But this does not hinder an approximate calculation of the effect (see, e.g., [232]).

Another interesting fact is that in all cases of adhesion of a turbulent jet to the surface of a body the further flow is determined by the laws of the semilimited turbulent jet. In particular, the velocity distributions in the cross sections of such jets agree virtually with the universal distribution given above (see Fig. 12.6).

The jet flow around bodies, the interaction between jets and many related problems are of great significance in practice, for the active control by processes occurring in the boundary layer (improved flow around a body, prevention of break-off, etc. see [232]), and also for the development of effective procedures of furnace devices, burners and stabilizers, ventilation devices, etc., and finally of highly-economical heat exchangers, drying and other apparatus which use, for example, the potential jet flow around badly streamlined bodies of the type of a cylinder with sharp reduction of drag (see Fig. 16.11).

REFERENCES

9, 11, 30, 36, 37, 45, 46, 51, 58, 59, 60, 76, 81, 86, 107, 108, 109, 110, 111, 112, 113, 114, 125, 126, 127, 128, 130, 136, 137, 138, 139, 140, 141, 147, 148, 158, 174, 179, 180, 185, 193, 194, 205, 206, 207, 209, 212, 219, 222, 232, 248, 291, 292, 293, 306, 307, 309, 318.

Manuscript Page No.	[Footnotes]
411	As this effect is of dynamic nature.

Chapter 17

THE AERODYNAMICS OF A GAS TORCH

17.1. THE DIFFUSION TORCH

One of the main applications of the theory of compressible gas jets is the problem of the applied theory of combustion, first of all the aerodynamics of a burning gas jet.

As always in the problems of the gasdynamics of combustion [67, 70, 114], we must distinguish between two statements of problem: the gasdynamic formulation, where the chemical reactions are assumed to take place at an infinitely high rate, and the stability problem, the theory of the thermal conditions of combustion, taking the kinetics into account.

Applied to the gas torch the first way of considering the problem is connected with the assumption that the process of combustion is localized to very narrow, in the limiting case infinitesimally thin zones, the so-called flame front. Outside these zones, in the entire surrounding space, the processes of momentum, heat and mass transfer are the only ones that take place.

We shall here restrict ourselves to this statement of problem of gas combustion, still reducing it further by additional assumptions. In the following we shall only consider the straight jet flame of gases not mixed beforehand, which is produced when one of the components of the combustible mixture (usually the fuel gas) issues in an unlimited space filled with the other component (usually the oxidizing agent,

e.g., air).

As regards the second limiting case of the gas torch, the combustion of a premixed uniform gas mixture, we shall not consider it here. In this connection, we restrict ourselves to the following remarks.

Firstly, it must be noted that the intense torchlike combustion of a homogeneous gas mixture, just as in the case of the burning of separate components, takes place in the flame front. In this way, as shown by G.N. Abramovich [4, 11], also this field is well accessible to the applied theory and methods of jet calculation.

Secondly, and this prompted us to renounce in this book a consideration of the aerodynamics of the homogeneous torch, in order to obtain a closed system of transfer equations in a homogeneous torch we must have an additional kinetic condition determining the position of the flame front.

The unsatisfactory state of the modern theory of propagation of turbulent combustion does not enable us to give this condition in a sufficiently effective form. Any detailed discussion of this problem is not within the framework of the present book.

In contrast to this, in the case of the so-called diffusion torch of nonmixed gases, as shown in a series of papers [58,62,67,70,95,97, 99,100,103], we have a closed system of transfer equations which, together with the boundary conditions, permit the determination of not only the distributions of velocity, temperature and of the concentrations of primary agents and combustion products, but also the position of the flame front without any additional hypotheses.

In the theory of diffusion combustion of gas one of the first papers published in this plan was the paper by S. Brooke and T. Schuman [234]. These authors considered the combustion of gas under the condi-

tions of parallel motion at the same velocities of uniform (plane or axisymmetric) streams of nonmixed components of the combustible mixture: fuel plus oxidizer. The one stream is considered to be surrounded by the other. The surface of combustion (the flame front) appearing in the zone of gas mixing, assumes a closed form (with respect to the axis) when the component moving in the center of the burner is supplied in an insufficient (compared to the stoichiometric) quantity and it is open in the opposite case. For the regions lying on both sides of the flame front the equations of molecular diffusion were integrated in this paper for the mixture of the respective component and the combustion products. The results obtained were "joined" at the flame front which was determined from the condition of zero concentration of reagents at the front and stoichiometrical ratio of the diffusion fluxes of the components streaming to the front. The velocity of motion of the gases, as already mentioned, was taken to be the same, the flow was considered free from turbulence. The authors therefore had to solve the simple equations of unsteady diffusion in the coordinates $t = \frac{x}{v_0}$, y where v_0 is the velocity (without sources as the combustion was assumed localized in the front). The influence of the aerodynamics of the flow and the temperature distribution (the diffusion constant and other quantities were assumed constant) were completely ignored in this case. The solution obtained and its experimental verification were at this time significant in principle for the development of the ideas on the diffusion combustion.

A more advanced statement of the problem of diffusion combustion of gases must comprise an integration of the whole system of equations, the equations of motion and continuity, energy and diffusion, with the corresponding boundary conditions. The main part is here played by the mixing process of gases which, naturally, is determined by the aerodyna-

mics of the flow. The developed procedure must apply to both laminar and turbulent gas combustion under the assumption of an infinitely high rate of the combustion reactions. Attempts of calculations of this kind were made repeatedly, by various authors [11,97,103,208, 228,290]. They all based their considerations in some way on the idea that the mixing plays a decisive part. "Burning — a consequence of mixing," this thesis contains in essential the whole real state of the theory of diffusion combustion.

But the development of a full value procedure of calculation was impeded by the lack of sufficient information on the processes of gas mixing. An essential progress may be achieved by the application of the methods of calculation developed in the theory of jets.

According to the order of treatment chosen in this book, we shall first consider the laminar diffusion combustion by way of the example of the combustion in the mixing zone of two parallel gas wakes, fuel and oxidizer, moving at different velocities (nonperturbed flow). As the second example of application of the methods of jet theory to problems of the aerodynamic theory of the torch we consider the combustion of a gas in the flame front of an axisymmetric turbulent diffusion torch. In order to solve this problem, we again use the method of the equivalent problem of the theory of thermal conductivity.

For the sake of simplicity and convenience, the description of effects will be somewhat simplified in both examples. In particular, though this is not a principle of the torch model investigated, we shall not take into account the effects of thermo- and barodiffusion, the variation of physical constants including specific heat in the reaction process, etc. This approximative statement of problem is approached more closely by the combustion of components diluted by an inert gas (combustion in air). It may also be used as a qualitative

picture of the reaction taking place in a highly exothermal binary gas mixture (e.g., oxygen and hydrogen).

17.2. LAMINAR DIFFUSION COMBUSTION

Let us consider the following problem (see diagram of Fig. 17.1). Imagine that along a plate, on either side of it the gas flows, i.e., the components of the combustible mixture, move along (without friction) with different values of the velocities. Beginning at point O, where the plate has its end, the region of gas mixing is formed which contains the region of steady burning. In the calculation, the gases will be considered to be compressible with equal and constant physical properties, except for the viscosity. The viscosity is assumed to be a linear function of the temperature. We shall also neglect the pressure change in the flame front owing to heat release and change of the number of moles in reaction. All these simplifications, as also the neglection of thermal diffusion, etc., mentioned before, do not cause in principle distortions in the qualitative pattern of the effect. When they are taken into account for a binary mixture, the expressions obtained are more cumbersome. As regards the chemical reaction rate, it is taken to be so high that, in a schematic representation of the effect, the combustion zone may be represented in form of a mathematical surface, namely the flame front. The supposition is fundamental in the diffusion theory of gas combustion. Experiments show that it is fully sufficient for an intense, evolved combustion of gases. On the other hand, in such cases where this assumption is no longer justified, in general doubt may be cast on the stability of intense combustion.

In order to solve the problem (carried out by Sh.A. Yershin and L.P. Yarin) we use the method described in detail in Chapter 8 where it was applied to laminar gas jets. In particular, for both problems (inert mixing and combustion) the solution of the dynamic problem on

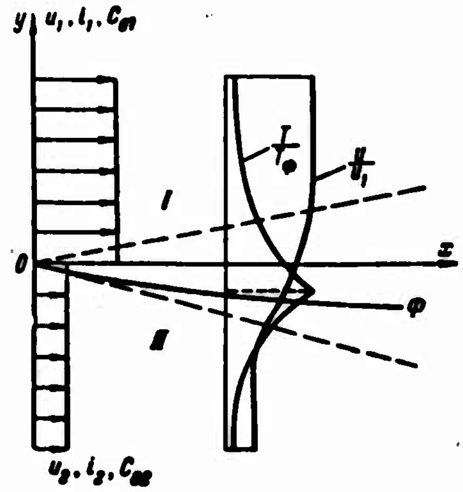

Fig. 17.1. Schematic representation of diffusion gas torch in the mixing zone of two parallel flows. I – Fuel and combustion products; II – oxidizer and combustion products; f – f flame front.

the mixing of parallel gas flows maintains its form. We shall therefore consider the way of solution only briefly, for details which are not specific of combustion we refer the reader to Section 8.6.

The initial equations transformed to the plane of A.A. Dorodnitsyn's variables are written in nondimensional form (using the denotations given below and omitting for simplicity the bar on the nondimensional quantities: $\bar{x}, \bar{\eta}, \bar{u}$ etc., see below).

To these equations belong (see Eq. 8.70) for $M \approx 0$):

$$u\frac{\partial u}{\partial \xi}+\tilde{v}\frac{\partial u}{\partial \eta}=\frac{\partial^2 u}{\partial \eta^2},\quad \frac{\partial u}{\partial \xi}+\frac{\partial \tilde{v}}{\partial \eta}=0 \qquad (17.1)$$

the equations of motion and continuity,

$$u\frac{\partial i}{\partial \xi}+\tilde{v}\frac{\partial i}{\partial \eta}=\frac{1}{\mathrm{Pr}}\frac{\partial^2 i}{\partial \eta^2} \qquad (17.2)$$

the energy equation, and

$$u\frac{\partial c_j}{\partial \xi}+\tilde{v}\frac{\partial c_j}{\partial \eta}=\frac{1}{\mathrm{Pr}_{\text{диф}}}\frac{\partial^2 c_j}{\partial \eta^2},\ j=1,2, \qquad (17.3)$$

the diffusion equations (for each component).

The denotations are defined as follows (we give the new and the old ones as well, the latter being repeated for convenience):

$$\bar{u} = \frac{u}{u_1 + u_2}; \ \bar{\rho} = \frac{\rho}{\rho_1}; \ \tilde{v} = \bar{u}\frac{\partial\bar{\eta}}{\partial\bar{x}} + \bar{\rho}\bar{v}; \ \bar{v} = \frac{2v}{u_1 + u_2}\sqrt{\mathrm{Re}}\sqrt{\frac{1+m}{2}};$$

$$m = \frac{u_2}{u_1}; \ \mathrm{Re} = \frac{u_1 l}{\nu}; \ \bar{\xi} = \frac{x}{l} \equiv \frac{\xi}{l}; \ \bar{y} = \frac{y}{l}\sqrt{\mathrm{Re}}\sqrt{\frac{1+m}{2}}; \ \bar{i} = \frac{i - i_j}{i_\Phi - i_j};$$

$$c_j = \frac{c}{c_{j0}}; \ \omega_j = \frac{\rho_j}{\rho_\Phi} = \frac{T_\Phi}{T_j} \ (j = 1, 2), \ i = c_p T; \ \mathrm{Pr} = \frac{\nu}{a};$$

$$\mathrm{Pr}_{\text{диф}} = \frac{\nu}{D};$$

$\xi = x; \ \eta = \int_0^y \bar{\rho}\,dy$ (ξ, η are Dorodnitsyn's variables, l is a characteristic length (eliminated in what follows). Subscripts: «Φ» refers to the flame front, 1 and 2 (or j, where j = 1,2) denote the given values of the variables (apart from the concentration of the components) in the incoming flows and at infinity for the first (fuel) and second (oxidizer) regions, respectively. For the concentrations c_j, $j = 1,2$ are variable values, c_{1_0} and c_{2_0} are given values for the same regions.

The boundary conditions for the velocity of flow will be the same as in the case of no combustion and mixing of inert gases. For the enthalpy and the concentrations of the components, besides the conditions at infinity we also give conditions for the flame front, which are fundamental for the problem considered and analogous problems. For the enthalpy this condition will be its constancy in the front where its value is maximum for combustion (in the general case, radiation and other losses are taken into account which are here ignored). For the reagent's concentrations the condition in the front surface (with $\xi = \xi_\Phi$, $\eta = \eta_\Phi$) is that they are equal to zero, firstly, and that in the diffusion flows reaching the front the ratio is stoichiometric, secondly. The use of these conditions together with the others enables us to solve the problem finally and not only to calculate the velocity, enthalpy and gas concentration distributions (of the components and the combustion products) in the whole field of flow, but also to determine the position of the

flame front.

The boundary conditions for nondimensional variables (as in the equations we omitted the bars) can be written in the form

$$\left.\begin{array}{l} \left.\begin{array}{l} u=\frac{1}{1+m}, \quad i=0,\ c_1=1 \text{ with } \eta=\infty, \\ u=\frac{m}{1+m}; \quad i=0,\ c_2=1 \text{ with } \eta=-\infty, \end{array}\right\}\xi>0, \\ i_\phi=1,\ c_{j\phi}=0,\ -D_1\frac{\partial c_1}{\partial n}\Big|_\phi=D_2\frac{\partial c_2}{\partial n}\Big|_\phi\Omega \text{ with } \xi=\xi_\phi;\ \eta=\eta_\phi, \end{array}\right\} \quad (17.4)$$

where Ω is the stoichiometric reaction coefficient.

Note that among the given parameters of the problem whose variation and its influence on the mathematical results is of greatest interest, there are two: the ratio of the flow velocities $m=\frac{u_2}{u_1}$ and the ratio of the gas densities, initial and behind the front: $\omega_j=\frac{\rho_j}{\rho_\phi}=\frac{T_\phi}{T_j}$, which characterize the thermal effect of the reaction (approximately $\omega_j=\frac{T_\phi}{T_j}=\frac{T_j+\frac{Q}{c_p}}{T_j}=1+\frac{Q}{c_pT_j}$, where Q is the thermal effect per unit of the mixture). The influence of these parameters will be shown by means of the calculation examples given below. It is essential that, qualitatively, it is the same also in the case of a turbulent diffusion torch. Let us now turn to the solution of the problem.

Since the latter is self-similar in the plane of A.A. Dorodnitsyn's variables, we introduce a generalized variable, $\varphi_D=\frac{\eta}{\sqrt{\xi}}$ and seek the solution in the form

$$u=F'(\varphi_D),\ i=\theta_j(\varphi_D),\ c_j=\pi_j(\varphi_D),\ j=1,2, \quad (17.5)$$

where the functions F, θ_j and π_j, in the same approximation as in Chapter 8, are determined by the usual differential equations of the form

$$\left.\begin{array}{r} F'''(\varphi_D)+\frac{1}{2}F(\varphi_D)F''(\varphi_D)=0, \\ \theta''(\varphi_D)+\frac{Pr}{2}F(\varphi_D)\theta'(\varphi_D)=0, \\ \pi''(\varphi_D)+\frac{Pr_{\text{диф}}}{2}F(\varphi_D)\pi'(\varphi_D)=0 \end{array}\right\} \quad (17.6)$$

with the boundary conditions $\operatorname{erf}(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt$.

$$\left.\begin{array}{llll} F' = \frac{1}{m+1}, & F'' = 0, \quad \theta_1 = 0, & \pi_1 = 1 & \text{with } \varphi_D = \infty, \\ F' = \frac{m}{m+1}, & \theta_2 = 0, \quad \pi_2 = 1 & & \text{with } \varphi_D = -\infty; \\ \theta_\Phi = 1, & \pi_{j\Phi} = 0, \quad j = 1, 2, & & \text{with } \varphi_D = \varphi_{D\Phi}. \end{array}\right\} \quad (17.7)$$

As also in Chapter 8 (and yet earlier, in Section (5.3) the solution of the equations is represented by expressions. Let us write these solutions in terms of dimensional variables:

$$\frac{u - u_2}{u_1 - u_2} = \frac{1}{2}\left[1 + \operatorname{erf}(\varphi_D)\right], \quad (17.8)$$

$$\frac{i - i_1}{i_\Phi - i_1} = \frac{1 - \operatorname{erf}(\varphi_D \sqrt{Pr})}{1 - \operatorname{erf}(\varphi_{D\Phi} \sqrt{Pr})}, \quad \frac{i - i_2}{i_\Phi - i_2} = \frac{1 + \operatorname{erf}(\varphi_D \sqrt{Pr})}{1 + \operatorname{erf}(\varphi_{D\Phi} \sqrt{Pr})}, \quad (17.9)$$

$$\frac{c_1}{c_{10}} = 1 - \frac{1 - \operatorname{erf}(\varphi_D \sqrt{Pr_{диф}})}{1 - \operatorname{erf}(\varphi_{D\Phi} \sqrt{Pr_{диф}})}, \quad \frac{c_2}{c_{20}} = 1 + \frac{1 + \operatorname{erf}(\varphi_D \sqrt{Pr_{диф}})}{1 + \operatorname{erf}(\varphi_{D\Phi} \sqrt{Pr_{диф}})}. \quad (17.10)$$

The equations (17.8)-(17.10) include the value of the coordinate $\varphi_{D\Phi}$ the flame front, which has not been determined hitherto.(When formulating these solutions and determining the constants of integration we did not make use of the condition defining the ratio of the components' flows at the front.)

Equation (17.8) describes the velocity variation in the whole field of flow. Equations (17.9) and (17.10) give the variation of enthalpy and concentration, respectively and refer to both the region of fuel (equations on lefthand side) and that of the oxidizer (righthand side equations). The influence of the various parameters on the real distributions of flow velocity, temperature (enthalpy) and concentrations will be explained in detail after returning to the plane of the physical coordinates.

For the return from the plane of A.A. Dorodnitsyn's variables to the plane of flow we apply, as usually, the expressions for the relations between the variables

$$x = \bar{\xi},\ \bar{y} = \int_0^{\bar{\eta}} \frac{d\bar{\eta}}{\bar{\rho}},\ \varphi = \frac{\bar{y}}{\sqrt{\bar{x}}} = \int_0^{\varphi_D} \frac{d\varphi_D}{\bar{\rho}(\varphi_D)}. \qquad (17.11)$$

Since the expressions for the density distributions in the jet cross section are different for the two zones, the final formulas of calculation linking the values of the nondimensional coordinates $\varphi = \frac{\bar{y}}{\sqrt{\bar{x}}}$ and $\varphi_D = \frac{\bar{\eta}}{\sqrt{\bar{\xi}}}$, will be written separately for the two regions:

$$\varphi = \varphi_D + \frac{\omega_1 - 1}{1 - \operatorname{erf}(\varphi_{D\Phi}\sqrt{Pr})}\Big\{\varphi_D - \Big[\varphi_D \operatorname{erf}(\varphi_D\sqrt{Pr}) + \frac{1}{\sqrt{\pi Pr}}(\exp(-\varphi_D^2 Pr) - 1)\Big]\Big\} \qquad (17.12)$$

with $\varphi_{D\Phi} < \varphi_D < \infty$,

$$\varphi = \varphi_\Phi + (\varphi_D - \varphi_{D\Phi}) + \frac{\omega_2 - 1}{1 + \operatorname{erf}(\varphi_{D\Phi}\sqrt{Pr})}\Big\{(\varphi_D - \varphi_{D\Phi}) + [\varphi_D \operatorname{erf}(\varphi_D\sqrt{Pr}) - \varphi_{D\Phi}\operatorname{erf}(\varphi_{D\Phi}\sqrt{Pr}) + \frac{1}{\sqrt{\pi Pr}}(\exp(-\varphi_D^2 Pr) - \exp(-\varphi_{D\Phi}^2 Pr))]\Big\} \qquad (17.13)$$

with $-\infty < \varphi_D < \varphi_{D\Phi}$

(here, as usually for the combustion in air or other typical combustible mixture, we assumed $\varphi_\Phi < 0$). The coordinate of the flame front φ_Φ is determined analogously as the quantity $\varphi_{D\Phi}$ from the equation

$$\varphi_\Phi = \varphi_{D\Phi} + \frac{\omega_2 - 1}{1 + \operatorname{erf}(\varphi_{D\Phi}\sqrt{Pr})}\Big\{\varphi_{D\Phi} + \Big[\varphi_{D\Phi}\operatorname{erf}(\varphi_{D\Phi}\sqrt{Pr}) + \frac{1}{\sqrt{\pi Pr}}(\exp(-\varphi_{D\Phi}^2 Pr) - 1)\Big]\Big\}. \qquad (17.14)$$

The value of $\varphi_{D\Phi}$ depends in its turn on the values of the parameters chosen in the calculation through the front conditions $-D_2 \frac{\partial c_2}{\partial n}\Big|_\Phi = \Omega D_1 \frac{\partial c_1}{\partial n}\Big|_\Phi$, which, taking the above solutions into account, assumes the form

$$\operatorname{erf}(\varphi_{D\Phi}\sqrt{Pr}) = \frac{1-\beta}{1+\beta},$$

where $\beta = \Omega \frac{c_{10}}{c_{20}} \frac{D_1}{D_2}$.

In this way, the solution of the problem has been finished.

In Fig. 17.2 we show examples of distributions of surplus temperature and velocity for three values of the compressibility parameter ω for one and the same value of $m = 0$ (one of the gases, the

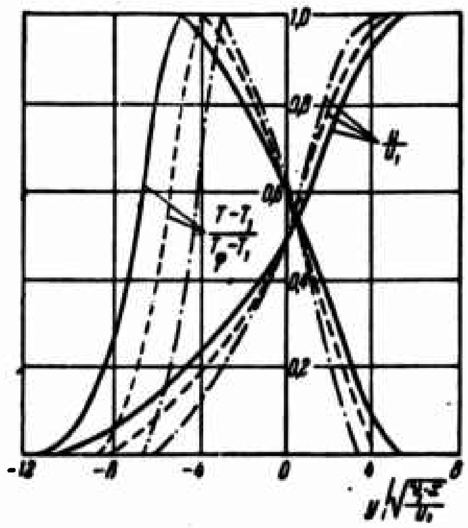

Fig. 17.2. Velocity and temperature distributions in laminar diffusion combustion of gas. $\beta = 4$; $Pr = Pr_{\text{диф}} = 1$; $m = 0$; $T_1 = T_2$; —·—·— $\omega = 6$, — — — $\omega = 8$, ——— $\omega = 10$.

oxidizer, is at rest). In Fig. 17.3 the same data are used and the dynamic pressure distribution is shown for three values of the parameter m and the single value $\omega = 8$. The other data referring to these examples of calculation, are given in the texts to the figures.

As we see from Fig. 17.2, the increase in specific heat release in the front (i.e., the temperature ratio and the corresponding density ratio $\omega_j = \frac{T_\phi}{T_j} = \frac{\rho_j}{\rho_\phi}$) results in a shift of the flame front toward the resting gas; in this case the velocity distribution becomes less steep.

We see from Fig. 17.3 that the displacement of the flame front position in the opposite direction occurs when the velocity values approach one another (when the value of m increases).

As already mentioned, these qualitative results which indicate the influence of the fundamental parameters on the position of the combustion front and on the form of the distributions of the character-

istic quantities also hold true for turbulent combustion. As a curious detail of the effect (which also refers to turbulent diffusion combustion) we mention the shape of the ρu^2 distribution curves in the cross sections of the burning jet. As we see from Fig. 17.3, for values of m which differ essentially from zero, a peculiar "well" appears in the ρu^2 distribution, which is caused by the fact that a sudden density drop takes place at the front, which is due to the heat release, the elevated temperature and the like. The front surface plays the part of something like a density "sink". For $m = 0$ this well is practically insignificant since the front lies in a zone of very low values of velocity (and ρu^2).

17.3 THE TURBULENT TORCH. CALCULATION ACCORDING TO THE METHOD OF THE EQUIVALENT PROBLEM

In spite of the clear statement of problem and the way of its solution which, in principle, is also clear, the development of the aerodynamic calculation of the turbulent diffusion gas torch is connected with a series of difficulties. In essential they were decisive for the imperfection of the methods of calculating turbulent gas jets. The calculations carried out by various authors were therefore extremely limited in the final results. Even when we do not mention the unidimensional calculation of jet and torch in paper [201], in investigations which are more accomplished from the aerodynamic point of view, the calculation also was reduced to a determination of the torch length and its position in the initial section of the jet. As shown in detail by Sh.A. Yershin [94,95] the "joining" of solutions which were obtained separately for the initial and the fundamental sections of the torch, did not yield satisfactory results. If, however, in torch calculations a more detailed jet model is used, with a division into three sections (initial, transient and fundamental), the calculation is so cumbersome

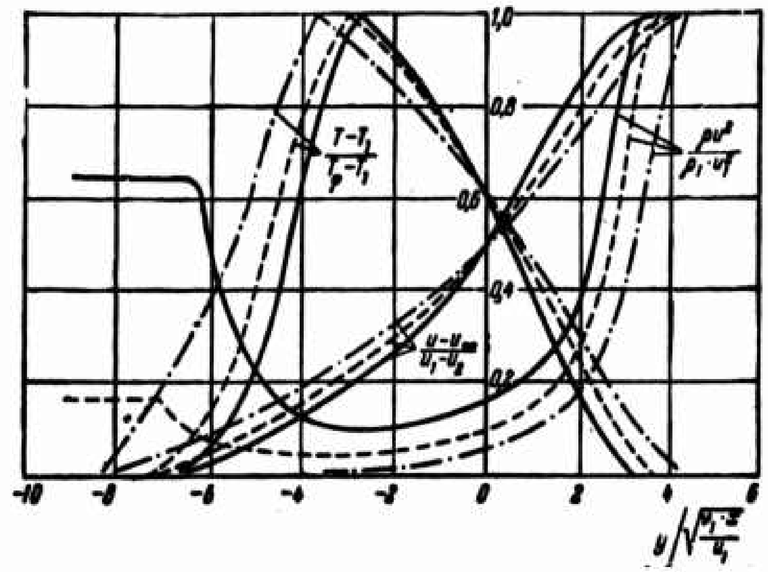

Fig. 17.3. Distributions of velocity, dynamic pressure and temperature in laminar diffusion combustion of gas. $\beta = 4$; $Pr = Pr_{диф} = 1$; $\omega = 8$; $T_1 = T_0$; —·—·— $m = 0$; — — — $m = 0{,}4$; ——— $m = 0{,}8$.

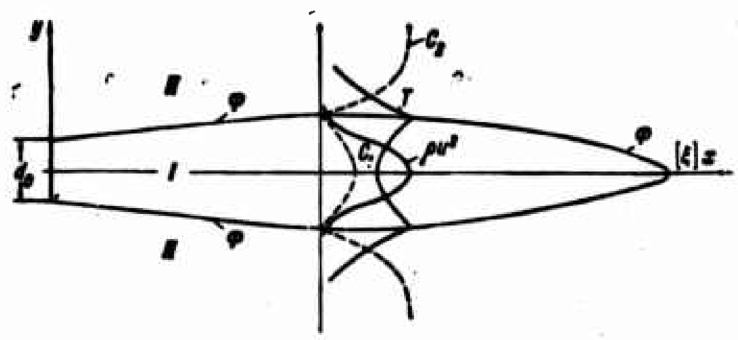

Fig. 17.4. Diagram of diffusion gas torch. I – Fuel plus combustion products; II – oxidizer plus combustion products; *f-f* flame front.

that it is not effective either.

Under these conditions it is most expedient (and this was proven by the first attempts made in the papers [94,228, etc.]) to use the method of the equivalent problem of the theory of thermal conductivity which enables us to calculate the continuous deformation of all distributions.

The most difficult and apriori unclear problem was here reduced to proving the applicability in the entire real space occupied by the torch and separated by the flame front into two zones (see diagram of Fig. 17.4) of one and the same form of transition to the linearized space, which was already used successfully when calculating the gas jets, namely the simplest form of transformation $\xi = \xi(x), \eta \approx y$. The generality of the turbulent transfer processes in gas jets without combustion and in the presence of combustion localized in the flame front, spoke in favor of this assumption. The decision, however, was reserved for experiment. As shown below, a comparison between the theoretical and experimental results proves satisfactory agreement. In this connection, we give a brief sketch of the course of the calculation.

Let us first formulate the physical statement of problem. The space occupied by the turbulent torch is divided into two zones according to the diagram of Fig. 17.4. In one of them (inner zone) fuel and combustion products are concentrated, the other (outer zone) contains air plus combustion products.

These zones are separated by the surface of the turbulent flame front averaged with respect to time. This surface is characterized by a maximum value of temperature, which is close to the theoretical value for complete combustion (losses taken into account). The calculation was based on the assumption that in this surface, and only in it, a practically instantaneous combustion takes place of the combustible

components. The fuel and oxidizer concentrations in the flame front are therefore considered to be equal to zero. The fluxes of fuel and oxidizer supplied by turbulent diffusion to the flame front where they burn are therefore in stoichiometrical equilibrium. In addition to the diffusion of substance to the flame front and away from it, turbulent heat transfer takes place in the torch on either side of the front, and also a fundamental process of dissipation of initial momentum flux which determines the entire course of the phenomenon.

Let us give a brief schematic representation of the torch calculation and, in the next section of the book, a comparison of theoretical and experimental results, following essentially the papers by Sh.A. Yershin and L.P. Yarin [98,99,100].

In order to calculate the momentum flux density distribution and the distributions of heat content and substance, we must integrate differential equations of the type of the heatconduction equation (for the axisymmetric problem)

$$\frac{\partial z_i}{\partial \xi_i} = \frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial z_i}{\partial y}\right) \quad (i = 1, 2, 3). \tag{17.15}$$

The subscript i suffixed to the function z and the coordinate ξ in Eq. (17.15) refer respectively to the quantities ρu^2 if $i = 1$, to $\rho u c_p \Delta T = \rho u c_p (T - T_\Phi)$ with $i = 2$ and finally, to $\rho u \Delta c = \rho u (c - c_\Phi)$ with $i = 3$. In the latter case ($i = 3$) we are concerned with the fuel concentration in the inner zone of the torch and that of the oxidizer on the outer zone. In both cases $\Delta c = c - c_\Phi = c$, since $c_\Phi \equiv 0$ for both the fuel and the oxidizer in the flame front surface.

Though, generally speaking, the coordinates ξ_i are different for different "substances" we assume that approximately $\xi_1 \cong 0,75\xi_2 \cong 0,75\xi_3$, i.e., the effective values of the turbulent Prandtl numbers are assumed to one another (the thermal and the diffusion Prandtl numbers) and equal

to about 0.75. In this case the only task of the experiment is the determination of the function $\xi_1 = \xi_1(x)$ which is obtained from a comparison of the calculated and experimental curves of variation of the quantity $(\rho u^2)_m$ on the jet axis.

The equations given above must be solved, as in the case of laminar combustion, separately for the outer and inner zone of the torch, such that the corresponding boundary conditions are satisfied on the jet axis, at infinity and in the interface, i.e., the flame front. The latter does not refer only to the equation for ρu^2 since the distribution of this quantity does not possess a small discontinuity at the front (as the temperature does).

In this way the dynamic problem is separated from the other problems and its solution is derived from the calculation of the submerged gas jet of finite dimensions (in the absence of combustion).

For the sake of completeness we give the boundary conditions.

I. In the inner zone of the torch (fuel plus combustion products)

$$\text{with } \xi = 0, \quad 0 < y < r_0 \quad z_i = z_{0i}$$

(in this zone $z_3 = \rho u c$, c being the fuel concentration).

II. In the outer zone (oxidizer plus combustion products)

$$\text{with } \xi = 0, \; r_0 < y < \infty \text{ and with } 0 < \xi < \infty, \; y \to \infty$$
$$z_1 \to 0, \; z_2 \to z_{\infty 2}; \; z_3 \to z_{\infty 3}$$

(here $z_3 = \rho u c$, c being the oxidizer concentration). In addition to this, in the flame front

$$\text{with } \xi = \xi_\phi, \; y = y_\phi \quad z_3 = z_3 = 0;$$
$$-D_I \frac{\partial c_I}{\partial n}\Big|_\phi = \Omega D_{II} \frac{\partial c_{II}}{\partial n}\Big|_\phi ,$$

where $\frac{\partial}{\partial n}$ denotes derivatives with respect to the normal to the flame front, the subscripts I and II refer to the inner (fuel) and the outer (oxidizer) zones, respectively. Moreover, from the condition of symmetry of the flow on the axis we have with $y = 0$ $z_1 = 0$.

We seek the solution of the system of equations (17.15) in the form of

$$z_i = a_i + b_i \Phi(\xi_i, y), \tag{17.16}$$

where the function

$$\Phi(\xi_i, y) = \frac{1}{2\pi\xi_i} \int_0^{r_0} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \exp\left(-\frac{y^2 + r^2 - 2yr\sin\theta}{4\xi_i}\right) r\,dr\,d\theta. \tag{17.17}$$

At the outermost point of the torch we have

$$\Phi = \Phi(\xi_{\phi\,\max}, 0) = 1 - \exp\left(-\frac{r_0^2}{4\xi_{\phi\,\max}}\right),$$

since the flame front is an isothermal surface.

Integral (17.17) is solved numerically. For an axisymmetric torch with uniform initial distributions it is convenient to use the so-called P-functions tabulated in a paper by J. Masters [264].

The solution of the system of equations (17.15), under these conditions, will read for the simplest case of constant and equal specific heats of the gases

$$\frac{\rho u^2}{(\rho u^2)_0} = \Phi(\xi_1, y), \tag{17.18}$$

$$\left.\begin{aligned} &\frac{T - T_0}{T_\phi - T_0} = 1 - \sqrt{\frac{\rho_0}{\rho}} F(\xi_2, y), \\ &\frac{c}{c_0} = \sqrt{\frac{\rho_0}{\rho}} F(\xi_2, y), \\ &\sqrt{\frac{\rho_0}{\rho}} = \frac{1}{2}\left\{-\left(\frac{\rho_0}{\rho_\phi} - 1\right) F(\xi_2, y) + \right. \\ &\qquad \left. + \sqrt{\left(\frac{\rho_0}{\rho_\phi} - 1\right)^2 F^2(\xi_2, y) + 4\frac{\rho_0}{\rho_\phi}}\right\}, \\ &F(\xi_2, y) = \frac{1}{\sqrt{\Phi(\xi_1, y)}}\left[1 - \frac{1 - \Phi(\xi_2, y)}{1 - \Phi(\xi_{2\phi}, y_\phi)}\right] \end{aligned}\right\} \text{ — for zone I} \tag{17.19}$$

$$
\left.
\begin{aligned}
&\frac{T-T_\infty}{T_\phi-T_\infty}=\sqrt{\frac{\rho_\phi}{\rho}}\,F_1(\xi_2,\,y),\\
&\frac{c}{c_\infty}=1-\sqrt{\frac{\rho_\phi}{\rho}}\,F_1(\xi_3,\,y),\\
&\sqrt{\frac{\rho_\phi}{\rho}}=\left(1-\frac{\rho_\phi}{\rho_\infty}\right)F_1(\xi_2,\,y)+\\
&\qquad+\sqrt{\left(1-\frac{\rho_\phi}{\rho_\infty}\right)^2F_1^2(\xi_2,\,y)+4\frac{\rho_\phi}{\rho_\infty}},\\
&F_1(\xi_2,\,y)=\frac{\Phi(\xi_2,\,y)}{\Phi(\xi_{2\phi},\,y_\phi)}\sqrt{\frac{\Phi(\xi_{1\phi},\,y_\phi)}{\Phi(\xi_1,\,y)}}
\end{aligned}
\right\}\text{— for zone II} \qquad (17.20)
$$

With given flame front coordinates ξ_ϕ, y_ϕ the solution obtained permits the construction of the velocity, temperature and concentration distributions in the torch cross sections.

When we differentiate the expression for the concentrations obtained and transform the result, we obtain an equation determining the coordinate $\xi_{\phi\,max}$ i.e., the length of the torch in the linearized space

$$
\frac{\exp\left(-\frac{r_0^2}{4\xi_{\phi\,max}}\right)}{\sqrt{1-\exp\left(-\frac{r_0^2}{4\xi_{\phi\,max}}\right)}}\left[\frac{\exp\left(\frac{r_0^2}{4\xi_{\phi\,max}}\right)-1}{\exp\left(\frac{r_0^2}{4\xi_{\phi\,max}}\mathrm{Pr}_T\right)-1}-\frac{1}{2\mathrm{Pr}_T}\right]=
$$
$$
=\frac{1}{2C_{\text{стех}}}\left\{\Omega c_{0\,\text{топл}}\sqrt{\frac{\rho_n}{\rho_\phi}}\left(1+\frac{\rho_\phi}{\rho_\infty}\right)\right\}. \qquad (17.21)
$$

The results of the calculation prove that the reduced torch length $\frac{\sqrt{\xi_{\phi\,max}}}{r_0}$ within a wide range of variation of the parameter Ω is a virtually linear function of this quantity. This means that for a given fuel that true torch length $l_\phi\sim\sqrt{\xi_{\phi\,max}}\sim r_0$ is proportional to the nozzle diameter, where the proportionality factor is determined as a whole by the stoichiometric conditions (by the same parameter).

This result is in agreement with the experiments of various authors conducted in order to determine the length of the evolved turbulent diffusion torch.

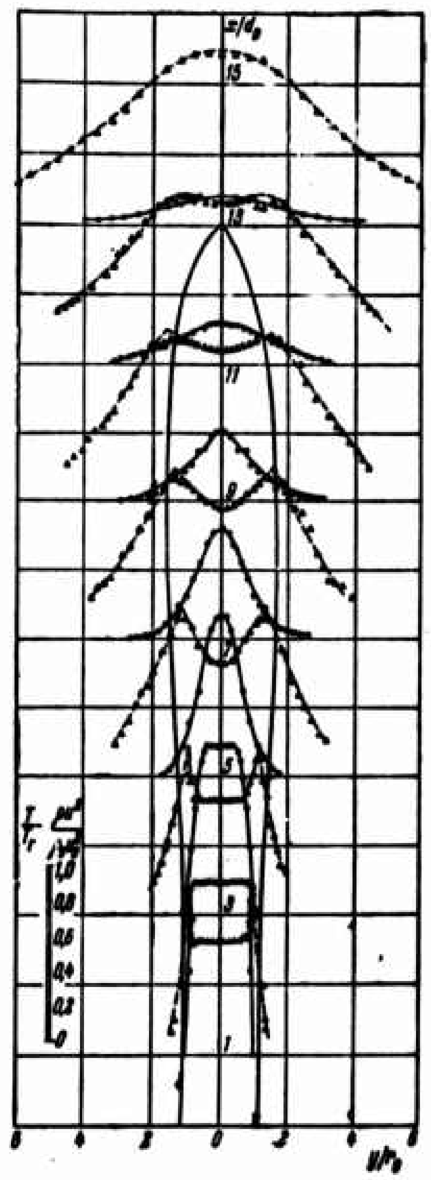

Fig. 17.5. Distribution curves of dynamic pressure (ooo) and temperature (ΔΔΔ) in torch cross sections (solid line: flame front surface) [100]

(T_0 = 1300°K, c_0 = 0,065 кг/кг, u_0 = 61 м/сек).

The calculation of a torch in the ξ, y space can thus be finished.

17.4. THE TURBULENT TORCH. COMPARISON WITH EXPERIMENT

By way of example we show in Fig. 17.5 the experimental ρu^2 and temperature distributions measured in several cross sections along a turbulent gas torch.

The middle parts of the graphs shown in Figs. 17.6-17.9 show the experimental and calculated (solid lines) distributions of ρu^2 and temperature in torch cross sections. As we see from these figures, the agreement between calculation and experiment, with the function $\xi(x)$ (see Fig. 17.7) chosen from experiment, is quite satisfactory.

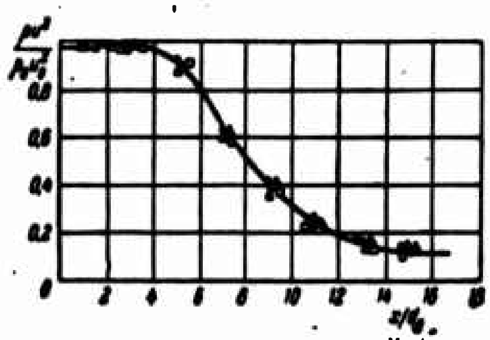

Fig. 17.6. Variation of dynamic pressure along the axis of a turbulent diffusion torch (u_0 = 40-70 m/sec, T_0 = 1100-1300°K, concentration of propane-butane gas c_0 = 0.053-0.120 kgf/kgf [100]).

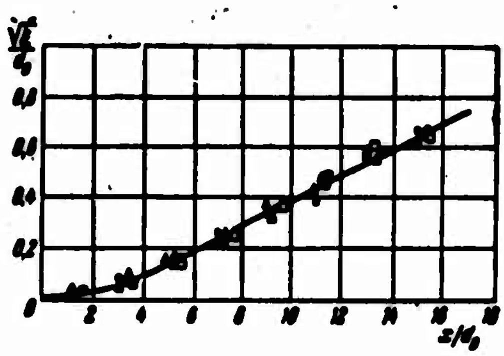

Fig. 17.7. Effective coordinate ξ as a function of the length of the gas torch [100].

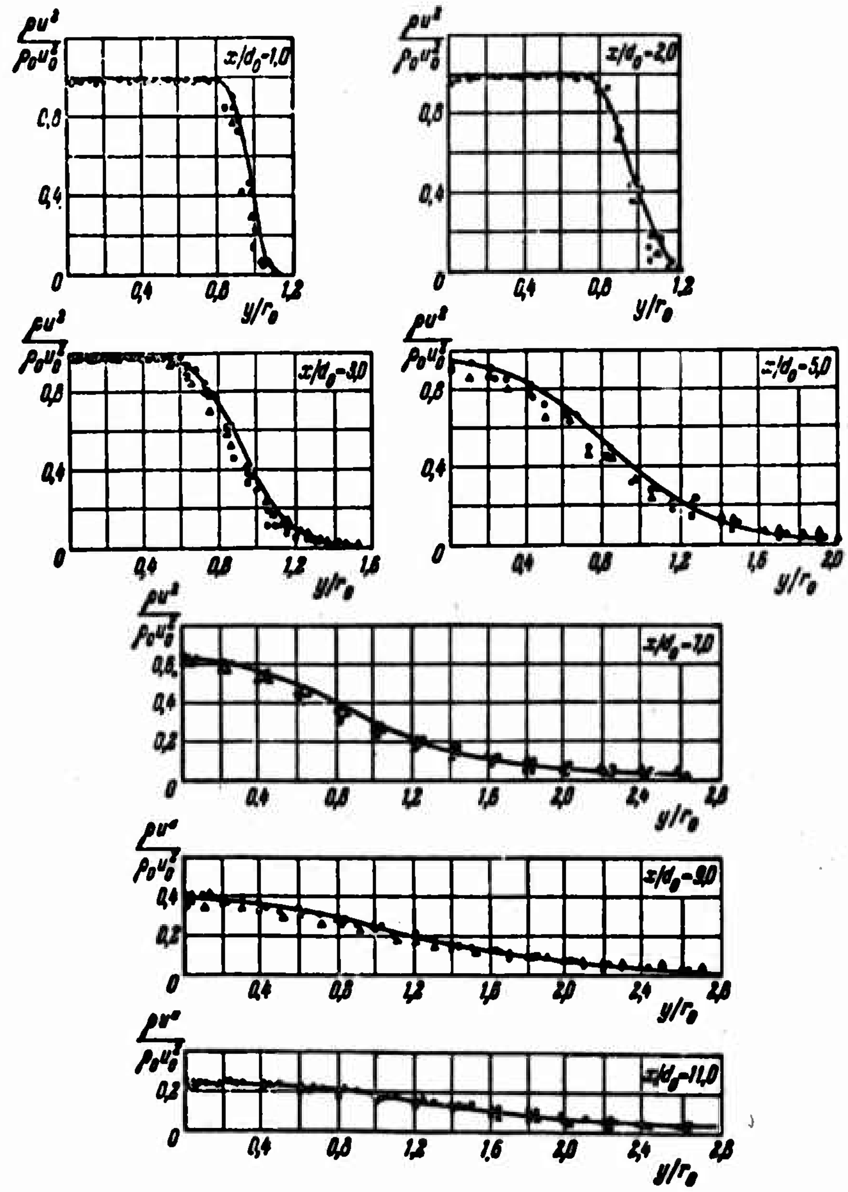

Fig. 17.8. Variation of ρu^2 in the cross sections of a turbulent diffusion torch (with different values of x/d_0). ——— calculation ○ ● △ ▲ □ experiment [100].

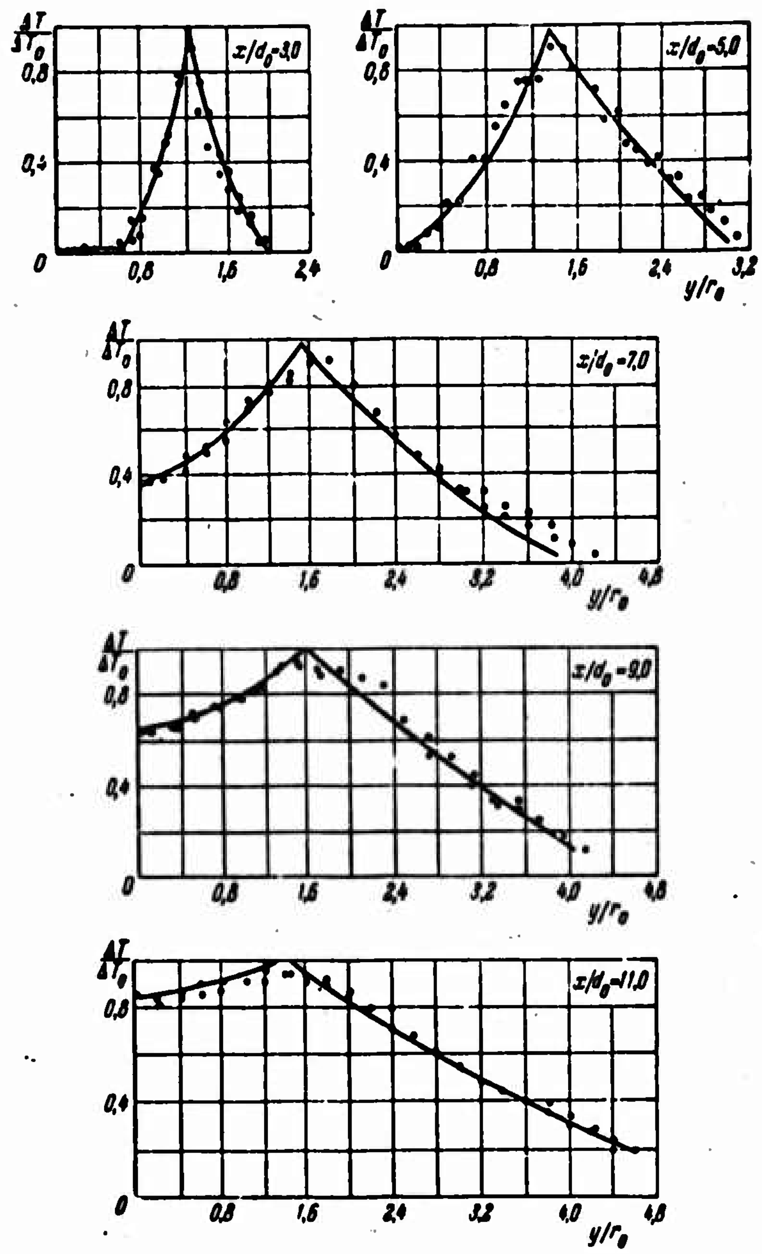

Fig. 17.9. Temperature distributions in the cross sections of a turbulent torch (with different values of x/d_0). ——— calculation; o – experiment [100].

In particular, as seen from Fig. 17.7, the transition from the linearized space to the real space, according to L.P. Yarin's data [100], is governed by the same relation between the variables as in the case of gas jets without combustion. For an approximate calculation it proved possible to neglect insignificant deviations from linearity in the very beginning of the torch and we can put $\sqrt{\xi_1} \approx \beta_1 x$, where $\beta_1 = 0{,}039 \div 0{,}043$, or, approximately, $\beta_1 \approx 0{,}04$. With this value in one of the experiments a torch length of $L/d_0 \approx 12$. was obtained.

In Fig. 17.10 this experiment corresponds to the middle curve of the torch contour. For it and two other experiments in the same figure we compared with one another the calculated lines of the flame front and the experimental points corresponding to the maximum temperature values in the torch cross sections.

A great number of analogous graphs of distrib tions of ρu^2, temperature, concentration and torch contour are given in the paper by L.P. Yarin [100] mentioned above.

The effectiveness of application of the method of the equivalent problem to the calculation of the diffusion torch is not unexpected according to what has been said above. But irrespectively of this, the possibility of precalculating satisfactorily the complete aerodynamic structure of a torch with so little empirical information is, in our opinion, very essential for the development of a theory and of mathematical methods of the aerodynamics of the torch.

From the physical point of view, this result proves the corroboration of the fundamental premises of the gasdynamic theory of the torch.

For an evolved tense turbulent torch it can be verified in this way that it is possible in the calculation to neglect the finite duration of the chemical reaction and also the influence of the coefficients of molecular transfer of the various gases*. Experiment and calculation

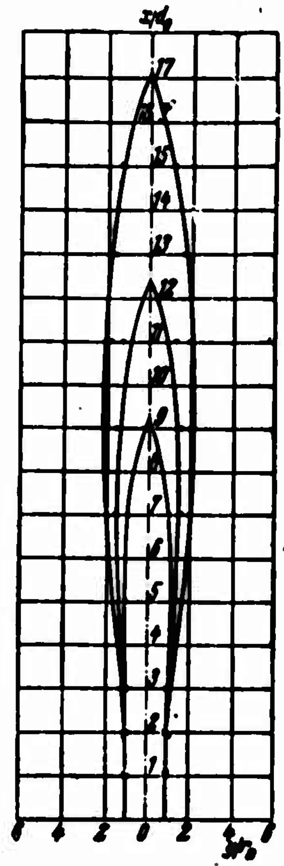

Fig. 17.10. Contours of flame front surfaces in a gas torch for various initial concentrations of the fuel [100] (propanebutane gas):

× — u_0 = 61 м/сек, T_0 = 1300°K, c_0 = 0,053 кг/кг; ○ — u_0 = 61 м/сек, T_0 = 1300°K, c_0 = 0,083 кг/кг; ● — u_0 = 61 м/сек, T_0 = 1200°K, c_0 = 0,12 кг/кг;

——— calculation according to the method of the equivalent problem (м/сек = m/sec; кг/кг = kg/kg)

also show that the mechanisms of turbulent transfer in gas jets without combustion and in the burning torch are the same.

Note that with the problem considered, which deals with the diffusion submerged torch, the flame front is very close to the effective boundary of the jet determined according to the dynamic pressure distribution (Fig. 17.8). As a consequence of this calculation one could have introduced a curve of variation of ρu^2 in the cross sections

which drops monotonically from the axis of the torch to its periphery. If the flame front were in the central part of the jet, for example in the zone of $\frac{\rho u^2}{(\rho u^2)_m} \gtrsim 0{,}5$, this character of the curve would be disturbed and one could observe a ρu^2 minimum near the front. This might be a consequence of a separation of density and velocity in the composite term ρu^2, as the flame front plays the part of a peculiar "well" of the density alone.

This case is encountered when considering a diffusion gas torch expanding in a parallel flow [99]. An evaluation of the experiments shows that when in the calculation a monotonically decreasing quantity, the surplus momentum, has been introduced in the form of $\rho u(u - u_\infty)$, and one formulates for it a heatconduction-type equation, etc., i.e., when for it the whole system of calculation according to the method of the equivalent problem is maintained, this yields good agreement between the theoretical and experimental results [154].

In this way, and this is very important, the mathematical procedure also applies to more complex problems (torch in parallel flow, etc.)

Finally, we must indicate that, starting from the aerodynamic model of the torch and completing it by the assumption of a finite rate of chemical reactions, it is possible, within the framework of a quasi-heterogeneous approximation, to treat the problems of flame front stability [67,220] and of the inherent completeness of combustion under given conditions. At the same time, taking the finite reaction rate into account (here not considered) one succeeds in obtaining all characteristics which are of interest in practice.

The results obtained in this section, which determine, in particular, the conditions of torch collapse, again prove the promising aspects of the jet model developed for the investigation of the process-

es of gas torch burning.

REFERENCES

11, 29, 42, 43, 44, 45, 58, 62, 67, 70, 72, 73, 81, 84, 94, 95, 97, 98, 99, 100, 102, 103, 114, 123, 124, 154, 164, 176, 182, 183, 184, 189, 190, 193, 201, 208, 220, 227, 228, 290.

Manu-script Page No.	[Footnotes]
435	It stands to reason that we are here concerned with a torch where the degree of combustion is very high, almost completely.

Chapter 18

JETS IN MAGNETOHYDRODYNAMICS

18.1. THE AXISYMMETRIC JET OF A CONDUCTIVE FLUID

In the last time one of the new promising fields of the mechanics of fluids and gases arouse great interest, namely the magnetohydro- and gasdynamics. It is therefore expedient to devote the last chapter of this book to a brief discussion of the theory of jets of viscous electrically conductive fluids expanding in a magnetic field. This discussion, which does not pretend to be comprehensive, restricts itself to describing the various types of MHD jet problems applying to the model of a quasi-neutral, continuous and isotropic medium moving with subrelativistic velocity.

Initially, following in essential paper [235], we consider the example of a magnetohydrodynamic generalization of the solutions of the Navier-Stokes equations completed by the corresponding equations of electrodynamics. We do this for the L.D. Landau problem on the expansion of a laminar axisymmetric jet of viscous conductive fluid in the presence of a magnetic field. Furthermore, we compare the solutions of several jet problems within the framework of the theory of the magnetic boundary layer (of first and second kind, according to the terminology by V.N. Zhigulev [101]) for high values of the magnetic Reynolds number Re_m. Finally we consider the problem on the expansion of jets of conductive fluids with low values of Re_m. Note that the problems chosen are not connected with any concrete applications of magneto-

hydrodynamics, but are of great theoretical interest in connection with an illustrative representation of the specific properties of jet motions of viscous conductive media.

Let us turn to the generalization of L.D. Landau's problem which was considered in detail in the first part of this book. Imagine a thin tube (see Fig. 18.1) being one of the poles of a magnet. The second pole, in the form of a tube of very large radius (theoretically $\frac{R_2}{R_1}\to\infty$) which is arranged coaxially with the first tube, is allowed to extend forwardly, in the direction of the flow, to an infinitely great distance from the source. The pattern of the magnetic lines of force corresponding to the aforesaid, is also shown schematically in Fig. 18.1.

Fig. 18.1. Diagram of the lines of force of the magnetic field. —— in the absence of motion; - - - with jet motion.

We now superimpose on this pattern the fluid flow produced by a source jet of viscous fluid issuing from the tube. As the magnetic parameter, we introduce the so-called magnetic Prandtl number $Pr_m = \frac{\nu}{\nu_m}$, where ν and $\nu_m = \frac{1}{\mu_m\sigma}$ are the coefficients of kinematic viscosity (the usual and the magnetic ones) μ_m and σ are the magnetic permeability and the electric conductivity, respectively, of the fluid*. Remember that the value of Pr_m is usually very small, approximately of the order of 10^{-4}-10^{-6}. The magnetic Prandtl number will therefore play the part of the small parameter in the problem considered. Owing to the smallness of Pr_m the value of the magnetic Reynolds number $Re_m = \frac{ud}{\nu_m}$ will also be small, even for a "strong" jet (see preceding part) for which the ordinary Reynolds number is great.

Owing to this fact, and to the fact that the magnetic lines of force and the streamlines of the fluid intersect essentially at small

angles (which are very small in the axial zone of flow, i.e., exactly where the velocity reaches its maximum values), the interaction between flow and field is weak in the problem discussed. In other words, it must be expected from general considerations that in a first approximation the character of the jet flow will differ only insignificantly from the analogous one, in the absence of a magnetic field. The fields with and without motion must also be very similar to one another. This qualitative result corresponds to the solution obtained in a paper by Cheng-Sheng Wu [235]. Let us discuss it briefly.

Having the material contained in the first part of the book at our disposal, the solution given below may be considered as a certain generalization of the solutions to the Navier-Stokes equations considered preciously for a jet of incompressible nonconductive fluid.

For the established motion of a viscous incompressible electrically conducting fluid the fundamental equations of continuity, motion (in addition to the electromagnetic volume force), the equations of induction and the condition that the magnetic field is free from sources, are written in a vector form:

$$\left.\begin{aligned} \operatorname{div} \boldsymbol{v} &= 0, \\ \operatorname{rot}\left\{\operatorname{rot} \boldsymbol{v} \times \boldsymbol{v} - \frac{\mu_m}{\rho} \operatorname{rot} \boldsymbol{H} \times \boldsymbol{H}\right\} &= \nu \Delta \operatorname{rot} \boldsymbol{v}, \\ \operatorname{rot}\{\nu_m \operatorname{rot} \boldsymbol{H} - \boldsymbol{v} \times \boldsymbol{H}\} &= 0, \\ \operatorname{div} \boldsymbol{H} &= 0. \end{aligned}\right\} \qquad (18.1)$$

Passing over to a spherical system of coordinates we assume, as in L.D. Landau's problem, for an axisymmetric self-similar motion

$$v_R = \nu \frac{F(\theta)}{R}, \quad v_\theta = \nu \frac{f(\theta)}{R} \qquad (18.2)$$

for the velocity component, and in the corresponding problem considered additionally

$$H_R = \nu \sqrt{\frac{\rho}{\mu_m}} \frac{H(\theta)}{R}, \quad H_\theta = \nu \sqrt{\frac{\rho}{\mu_m}} \frac{h(\theta)}{R} \qquad (18.3)$$

for the vector components of the magnetic field strength.

Since the vectors v and H are solenoidal, the nondimensional functions in Eqs. (18.2) and (18.3) satisfy the equations

$$\left.\begin{aligned} F(\theta) &= -f(\theta)\operatorname{ctg}\theta - f'(\theta), \\ H(\theta) &= -h(\theta)\operatorname{ctg}\theta - h'(\theta). \end{aligned}\right\} \qquad (18.4)$$

Integrating the equation of motion, taking Eq. (18.4) into account, we obtain after several transformations the ordinary differential equation

$$\Phi''(\theta) + 3\operatorname{ctg}\theta \cdot \Phi'(\theta) - 2\Phi'(\theta) = 0, \qquad (18.5)$$

where $\Phi(\theta)$ is a new function:

$$\Phi(\theta) \equiv f'(\theta) + \frac{1}{2}h^2(\theta) - \frac{1}{2}f^2(\theta) - f(\theta)\operatorname{ctg}\theta + c = \frac{a\cos\theta - b}{\sin^2\theta}.$$

From the induction equation we obtain analogously

$$h''(\theta) + [h(\theta)\cdot\operatorname{ctg}\theta]' = \mathrm{Pr}_m\,[f(\theta)h'(\theta) - f'(\theta)h(\theta)]. \qquad (18.6)$$

In this way, the problem is reduced to the integration of two ordinary differential equations (18.5) and (18.6) which, together with Eq. (18.4) with the correspondingly boundary conditions, determine the fields of v and H.

As in the case of a jet without magnetic field (see first part) a concrete motion can be separated when the values of the constants a, b, c in Eq. (18.5) are given. From the three cases given in paper [235]:

1) $a = b = c = 0$ generalized L.D. Landau problem [122],

2) $a = b = -2c$ generalized H. Squire problem [303],

3) $a = c = 0$, $b \neq 0$ generalized problem on the fan type jet, which correspond to the jet from a thin tube, the jet from a hole in a wall and the fan type jet, respectively, we consider for brevity only the first one.

The solutions of Eqs. (18.5) and (18.6) are sought in the form of series with respect to the small parameter Pr_m:

$$\left.\begin{aligned} f(\theta) &= f_0(\theta) + \mathrm{Pr}_m f_1(\theta) + \mathrm{Pr}_m^2 f_2(\theta) + \ldots, \\ h(\theta) &= h_0(\theta) + \mathrm{Pr}_m h_1(\theta) + \mathrm{Pr}_m^2 h_2(\theta) + \ldots \end{aligned}\right\} \qquad (18.7)$$

In these series the zero approximation corresponds to the flow of a nonconductive fluid. In this case, a magnetic field has no influence on the motion and the motion does not affect the field. In this case we have

$$\left.\begin{aligned} f_0' - \frac{1}{2} f_0^2 - f_0 \operatorname{ctg}\theta = 0, \\ h''_0 + (h_0 \operatorname{ctg}\theta)' = 0. \end{aligned}\right\} \qquad (18.8)$$

The solution of the first equation from (18.8) corresponding to L.D. Landau's problem was considered previously in detail. From the second equation, we obtain an expression for the vector compoments of the magnetic field strength in the absence of a jet (static field):

$$H_\theta(v = 0) = -\frac{\nu}{R}\sqrt{\frac{\rho}{\mu_m}}\, a_1 \frac{1-\cos\theta}{\sin^2\theta}, \quad H_{R0} = \frac{\nu}{R} a_1 \sqrt{\frac{\rho}{\mu_m}}. \qquad (18.9)$$

The equations of the first approximation where the interaction between motion and field is taken into account have the form

$$\left.\begin{aligned} f_1' - (f_0 + \operatorname{ctg}\theta) f_1 + h_0 h_1 = 0, \\ h_1'' + (h_1 \operatorname{ctg}\theta)' + f_0' h_0 - f_0 h'_0 = 0. \end{aligned}\right\} \qquad (18.10)$$

The solutions of this system of equations cannot be given in a simple closed form. We therefore give (in a series of graphs) the final results of calculations as obtained in paper [235] mentioned.

In Fig. 18.2, the solid lines represent the lines of force of the magnetic field and the streamlines of the nonconductive fluid (zero approximation) for one value of the parameter of the dynamic problem, L, for the "strong" jet ($L = 0.99$). The dashed lines in this figure represent the results of numerical integration of the equations in a first approximation (18.10). In this and the following figures the value of the magnetic Prandtl number is equal to $\mathrm{Pr}_m = 10^{-3}$, i.e., it is

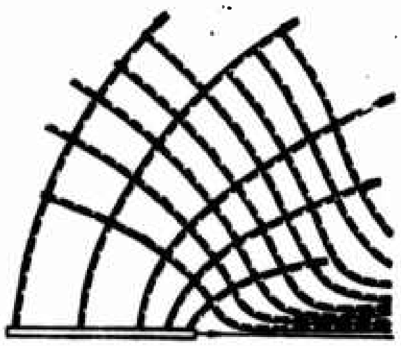

Fig. 18.2. Magnetic lines of force and streamlines in a "strong" jet
($L = 0{,}99$, $Pr_m = 10^{-?}$); [illegible]

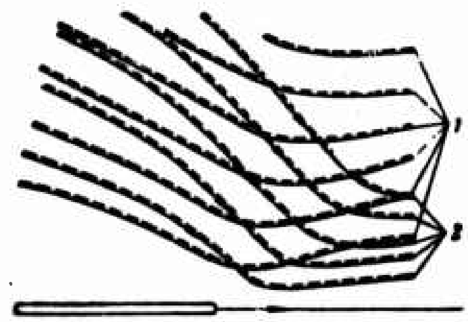

Fig. 18.3. Streamlines in a jet of conducting fluid
$Pr_m = 10^{-?}$; $1 - L = 0{,}5$; $2 - L = 0{,}01$ [illegible];
——— $H = 0$; — — — — $H \neq 0$.

relatively high. The corresponding mathematical formulas are not given as they are too voluminous.

In Figure 18.3, we give for comparison the results of an analogous calculation of the streamlines for two values of the jet parameter L.

As we see from the figures, in all cases the perturbation of the motion by the magnetic field and the perturbation of the field by the motion of the conductive fluid are insignificant in a first approximation. This result agrees with what has been said above.

In paper [235] the influence of a magnetic field on the temperature distribution in a jet is calculated in the same approximation (and also in the absence of an applied electric field). When we neglect the energy dissipation (Joulean and viscous), which corresponds to the general statement of problem, the differential equation of heat propagation is the same as that which corresponds to L.D. Landau's problem (nonconducting medium or absence of field). Under these conditions, the influence of the magnetic field on the temperature distribution

will be an indirect one, through the variation of velocity. The final expression for the temperature distribution will read

$$\frac{T-T_{\infty}}{(T-T_{\infty})_{\theta=0}} = \exp\left[-\Pr\int_0^{\theta}(\Pr_m f_1 + \Pr_m^2 f_2 + \ldots)\,d\theta\right]. \qquad (18.11)$$

We see from this that in a first approximation the deviation from the problem without interaction is small.

Analogously one calculates the influence of the magnetic field on the motion produced by a jet issuing from the opening in a plane wall or by a fan type jet (which is also insignificant in a first approximation).

18.2. THE EQUATIONS OF A MAGNETIC BOUNDARY LAYER

According to the terminology by V.N. Zhigulev [101], with high values of the magnetic Reynolds number Re_m, the region in which the influence of the magnetic field on the motion of the conducting fluid is concentrated principally will be called the magnetic boundary layer. Following paper [101] we shall also distinguish between two kinds of magnetic boundary layers.

In the first of them the magnetic field strength vector lies in one plane with the velocity vector and the angle made by them is small. As usually we may refer ourselves to a plane or an axisymmetric jet flow of incompressible conducting fluid. This case will be called the magnetic boundary layer of the first kind.

In the second case, the vector H is normal to both velocity components, i.e., it is directed along the z-axis in the plane of flow (or along the axis φ in the axisymmetric case). This layer will be called the magnetic boundary layer of second kind.

The magnetic boundary layer equations are obtained by the method of estimation, where, in addition to the usual estimation of velocity

components and coordinates ($v_x \sim 1$, $v_y \sim \delta$, $x \sim 1$, $y \sim \delta$, $\nu \sim \delta^2$) we assume for the vector components of the magnetic field strength $H_x \sim 1$, $H_y \sim \delta$, and also $\nu_m \sim \delta^2$.

Let us estimate the terms of the initial equations of magnetohydrodynamics [121,122] which we give for brevity in vector representation

$$\left.\begin{aligned}
\rho(v\nabla)v = -\nabla\tilde{p} + \mu_m(H\nabla)H + \operatorname{Div}\tau,\\
\operatorname{div} V = 0,\\
\operatorname{rot}(v \times H) = -\nu_m\Delta H,\\
\operatorname{div} H = 0,\\
\operatorname{div}\left(c_pTv - \frac{\lambda}{\rho}\operatorname{grad} T - v\operatorname{rot} v \times v\right) + \frac{\nu_m}{\rho}j^2 = 0.
\end{aligned}\right\} \qquad (18.12)$$

In the first of these equations and in the following we use the symbol $\tilde{p} = p + \frac{1}{2}\mu_m H^2$ to denote the sum of hydrostatic pressure p and magnetic pressure $p_m = \frac{1}{2}\mu_m H^2$ in the energy equation j denotes the electric current density.

Derivingthe equations of a plane steady magnetic boundary layer of first kind we assume the field H plane; $H(H_x, H_y, 0)$. After a corresponding estimation we obtain ($v_x \equiv u$, $v_y \equiv v$) :

$$\left.\begin{aligned}
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{1}{\rho}\frac{\partial\tilde{p}}{\partial x} + \frac{\mu_m}{\rho}\left(H_x\frac{\partial H_x}{\partial x} + H_y\frac{\partial H_x}{\partial y}\right),\\
\frac{\partial\tilde{p}}{\partial y} = 0 \quad \left(p + \frac{\mu_m H^2}{2} = \text{const}\right),
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = 0,\\
vH_x - uH_y = \nu_m\frac{\partial H_x}{\partial y},\\
c_p u\frac{\partial T}{\partial x} + c_p v\frac{\partial T}{\partial y} = \frac{\lambda}{\rho}\frac{\partial^2 T}{\partial y^2} + \nu\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\nu_m}{\rho}j^2.
\end{aligned}\right\} \qquad (18.13)$$

In order to derive analogous equations for a plane magnetic boundary layer of second kind we assume the magnetic field directed along the axis Oz: $H(0, 0, H_z)$. In this case (with $H_z = H$) we obtain by means of estimations

$$
\left.\begin{aligned}
&u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}-\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x},\\
&\frac{\partial \tilde{p}}{\partial y}=0\quad\left(p+\frac{1}{2}\mu_m H^2=\text{const}\right),\\
&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\\
&u\frac{\partial H}{\partial x}+v\frac{\partial H}{\partial y}=\nu_m\frac{\partial^2 H}{\partial y^2},\\
&c_p u\frac{\partial T}{\partial x}+c_p v\frac{\partial T}{\partial y}=\frac{\lambda}{\rho}\frac{\partial^2 T}{\partial y^2}+\nu\left(\frac{\partial u}{\partial y}\right)^2+\frac{\nu_m}{\rho}j^2.
\end{aligned}\right\}\qquad(18.14)
$$

In both cases of a plane magnetic boundary layer for a jet motion the total pressure $\tilde{p}$ is constant throughout the field of flow. The variation of the static pressure p is then determined by the law of variation of the magnetic field strength both along and across the boundary layer.

As also previously in the theory of the laminar boundary layer, we restrict ourselves to self-similar motions and study the form these equations will possess. For this purpose, we extend the exponential formulas of self-similarity transformations to the magnetic field strength.

For the magnetic boundary layer of first kind we put

$$
\left.\begin{aligned}
u&=Ax^{\alpha}F'(\varphi),\\
H_x&=Hx^{\omega}h'(\varphi),\\
\varphi&=Byx^{\beta},
\end{aligned}\right\}\qquad(18.15)
$$

so that from the condition of vanishing divergence of the velocity and magnetic field strength vectors we obtain

$$
v=-\frac{A}{B}x^{\alpha-\beta-1}[(\alpha-\beta)F+\beta\varphi F'],
$$

$$
H_y=-\frac{H}{B}x^{\omega-\beta-1}[(\omega-\beta)h+\beta\varphi h'].
$$

Calculating the derivatives and substituting them together with (18.15) in Eq. (18.13) we obtain a system of self-similar equations of the magnetic boundary layer of first kind in the form

$$F''' + \frac{A}{2\nu B^2}[(\alpha+1)FF'' - 2\alpha F'^2] = \frac{\mu_m}{\rho}\frac{H^2}{2\nu AB^2}[(\alpha+1)hh'' - 2\alpha h'^2], \quad (18.16)$$

$$h'' + \frac{A(\alpha+1)}{2\nu B^2}\,\mathrm{Pr}_m(Fh' - F'h) = 0 \quad (18.17)$$

with the condition

$$\alpha = \omega = 2\beta + 1. \quad (18.18)$$

linking the values of the constants of self-similarity.

As regards the equation of heat propagation, it cannot be reduced to a self-similar equation when Joulean heat (or the heat of friction) is taken into account.

Let us now derive analogous self-similar equations and the conditions linking the constants for the magnetic boundary layer of second kind:

$$F''' + \frac{A}{2\nu B^2}[(\alpha+1)FF'' - 2\alpha F'^2] = 0, \quad (18.19)$$

$$h'' + \frac{A}{2\nu B^2}\,\mathrm{Pr}_m[(\alpha+1)Fh' - 2\omega F'h] = 0. \quad (18.20)$$

Here, as always with a laminar boundary layer, $\alpha = 2\beta + 1$, but, unlike the case of the equations of the magnetic boundary layer of second kind, the constant ω of self-similarity (the exponent of E) is in no connection with the others.

The self-similar equations obtained will be used in the following sections in order to solve some concrete problems.

18.3. SOME REMARKS

For the purpose of a general judgment of the influence of a magnetic field on the jet flow of a viscous conducting fluid, we first consider for the case of the equations of the magnetic boundary layer of first kind the limiting case of a highly viscous high electric conductivity fluid for which $\mathrm{Pr}_m = \infty$. Note that this case is rather of theoretical interest since in technical applications (disregarding the field

of astrophysics) the magnetic Prandtl number is usually much smaller than unity.

Assuming $Pr_m = \infty$ we exclude at the same time from consideration Joulean dissipation and, owing to this, we are able to elucidate in a pure form the influence of the mechanical effect, the action of the electromagnetic volume force.

With $Pr_m = \infty$ Eq. (18.17) results in a similarity of the distributions of velocity and magnetic field strength and in a coincidence of the magnetic lines of force and the streamlines: $h' = F'$, $h = F$. This result is a particular case of the general theorem on the freezing of lines of force in a perfectly conducting fluid.

Using the conditions of similarity in Eq. (18.16) we give it the following form:

$$F''' + \frac{A}{2\nu B^3}(1 - \quad\quad + 1)FF'' - 2\alpha F'^2] = 0, \tag{18.21}$$

where the symbol S denotes the parameter of the problem

$$S = \frac{\mu_m H^2}{\rho A^2}. \tag{18.22}$$

It is easy to see that, introducing the new independent variable $\varphi = (1 - S)\varphi$ and the denotation $\bar{F} = F(\bar{\varphi})$, the latter equation can be transformed to

$$\bar{F}''' + \frac{A}{2\nu B^3}[(\alpha + 1)\bar{F}\bar{F}'' - 2\alpha\bar{F}'^2] = 0. \tag{18.23}$$

Equation (18.23) corresponds identically to the equation considered in Chapter 4 for a laminar plane boundary layer of conducting fluid. This indicates that for concrete self-similar jet motions (mixture of uniform flows, plane source jet, free and semilimited jets and the like) the expressions determining the nondimensional transverse velocity distributions for all values of the parameter $0 \leqslant S \leqslant 1$, will be one and the same:

$$\frac{u}{u_m} = F'(\varphi) = F'(\varphi)|_{S\to 0}. \qquad (18.24)$$

This function also determines the distribution of the longitudinal magnetic field component $\frac{H_x}{H_{xm}} = \frac{u}{u_m}$ in the jet cross section.

When we compare the u/u_m distribution as a function of the variable for various values of S, we see that the effective thickness of the jet rises as the parameter S is increased (with $S \to 1$ and $\frac{u}{u_m} \to 1$). As regards the velocity drop along the jet axis, it will also be the more intense the higher the value of the parameter S which is expressed by the value of the constant A in the formula $u_m = Ax^s$. With the highest admissible value of $S = 1$, we obtain $A = 0$, which corresponds to the limiting case of a degenerate jet (zero flow rate of fluid). The physically obvious condition of the existence of the jet $(G > 0)$ will be used as a criterion for the reality of the solutions consider[illegible] n the following.

The results here obtained for the plane motion also remain valid for the axisymmetric jet motion with the same condition of $Pr_m = \infty$.

To raise the illustrativeness we shall briefly consider one of the examples (plane source jet with $Pr_m = \infty$) in what follows, after a discussion of the more general problem of a plane source jet with a finite value of Pr_m. Other examples of solution (in the Mises variables) with $Pr_m = \infty$ are given on papers [105, 106].

We give without derivation the final formulas which determine the solutions of several particular cases and the values of the constants A, B, and H entering them.

In the case finite conductivity, i.e., when Pr_m is different from infinity., the distributions of velocity and magnetic field will not be similar, but with symmetrical boundary conditions a qualitative "analogy"

№ п/п	Вид движения [1]	$\frac{u}{u_m} = \frac{H_x}{H_{xm}}$	A	B	H
1	[2] Край плоской струи	$1+\frac{1}{2}(m-1)\times$ $\times(1-\mathrm{erf}\,\varphi)$	$(1-S)u_\infty$	$\frac{1}{2}\sqrt{\frac{(1-S)u_\infty}{\nu}}$	$S(1-S)\sqrt{\frac{\rho}{\mu_m}}$
2	[3] Плоская струя-источник	$\mathrm{ch}^{-2}\varphi$	$\frac{1}{2}\sqrt[3]{\frac{3(1-S)J^2_x}{4\rho^2\nu}}$	$\frac{1}{2}\sqrt[3]{\frac{J_x(1-S)^2}{6\rho^2\nu^2}}$	$\frac{1}{2}S\sqrt{\frac{\rho}{\mu_n}}\times$ $\times\sqrt[3]{\frac{3(1-S)J^2_x}{4\rho^2\nu}}$
3	[4] Плоская полуограниченная струя-источник	$F'=\frac{2}{3}\times$ $\times(F_\infty^{3/2}F^{1/2}-F^2)$	$\sqrt{\frac{K(1-S)}{4\rho^2\nu J_1}}$	$\frac{1}{2}\sqrt[4]{\frac{K(1-S)^3}{4\rho^2\nu^3 J_1}}$	$\sqrt{\frac{K(1-S)S}{4\rho\nu\mu_m J_1}}$
4	[5] Осесимметричная струя-источник	$\left(1+\frac{1}{8}\varphi^2\right)^{-2}$	$\frac{3J_x(1-S)}{8\pi\rho\nu}$	$\sqrt{\frac{3J_x(1-S)^2}{8\pi\rho\nu^2}}$	$\frac{3J_xS(1-S)}{8\pi\nu\sqrt{\rho\mu_m}}$

1) Form of motion; 2) edge of plane jet; 3) plane source jet; 4) plane semilimited source jet; 5) axisymmetric source jet.

of the u and H_x distributions still exists and also the character of the influence of the magnetic field (the parameter S) on the motion will be analogous. In particular, for flows with finite conductivity, the limitation of the values of the parameters S $0 \leqslant S \leqslant 1$ will remain in force. The physical meaning of the parameter S will be explained in the following.

All what has been said here refers to the magnetic boundary la of first kind. It is senseless to discuss the analogous case of for the magnetic boundary layer of second kind as the dynamic equ (18.19) does not contain any terms which are connected with the magn tic field (the magnetic field has no influence on the motion).

The fact that the dynamic problem is independent of the induction equation and the fact that Eq. (18.20) is solved after the determination of the velocity distribution indicates the direct analogy between the equations of the magnetic boundary layer of second kind and the sys-

tem of equations of the dynamic and the thermal boundary layers for a nonconducting fluid.

In addition to this, Eq. (18.20) is itself identical with the equation of heat propagation in a nonconducting fluid. The presence of this analogy, with similar boundary conditions for the temperature in the thermal problem and a magnetic field strength $H = H_z$ in the MHD problem permits the application of all solutions of the thermal problems compiled in Table 7.1 in order to describe the magnetic field distributions.

This curious "magneto-thermal" analogy, which also applies to the axisymmetric magnetic boundary layer of second kind, is conserved for all values of the magnetic Prandtl number Pr_m. The latter, as also the ordinary Prandtl number Pr in the thermal problem, characterizes the ratio between the effective thicknesses of the dynamic and the magnetic boundary layers.

This analogy is of course inapplicable to the magnetic boundary layer of first kind. The difference between them is due to the fact that only in the equation of the magnetic boundary layer of second kind the magnetic field strength plays the part of a scalar quantity.

In this case, the lines of force (which are directed parallel to the axis O_z in the plane of motion or in concentric circles lying in planes normal to the axis of symmetry of the round jet in the axisymmetric case) will neither be bent nor stretched. The only dynamic effect which is connected with the action of the magnetic field will be the appearance of a gradient of magnetic pressure $p_m = \frac{\mu_m H^2}{2}$ (in equilibrium with the hydrostatic pressure p).

As regards the influence of the magnetic field on the heat transfer (which appears when Eqs. (18.19), (18.20) and the energy equation are solved at a time), it will be considered by way of a concrete

example.

Note that, finally, the considerations of this section of the book (and to a certain degree those of the whole chapter dealing with MHD jets) may be transferred, at least qualitatively, to turbulent MHD jets of incompressible fluid. This applies particularly to the "magneto-thermal" analogy for the layer of second kind and to the limiting case of an infinitely large "turbulent magnetic Prandtl number" which is interesting for astrophysical applications (turbulent motions of perfectly conducting gas).

Remember in this connection that with the assumption of an asymptotic layer $\nu_T \sim x^{\alpha-\beta}$ the equations of self-similar flows and their solutions for laminar and turbulent jets agree with one another (see Tables 7.1 and 11.1).

A more complete discussion of these problems would be beyond the framework of the present book.

18.4. THE MAGNETIC BOUNDARY LAYER OF THE FIRST KIND

By way of example of the magnetic boundary layer of first kind we consider the problem of the propagation of a laminar plane source jet of incompressible conductive fluid, under the assumption that the magnetic Reynolds number Re_m is finite but high so that it is justified to apply the approximation of the magnetic boundary layer [315].

The initial equations for the problem on the plane source jet are Eqs. (18.16) and (18.17). The boundary conditions read

$$\left.\begin{aligned} F' = 1,\ F = h = 0 \ \text{with}\ \varphi = 0, \\ F' = h' = 0\ \text{with}\ \varphi = \pm\infty. \end{aligned}\right\} \qquad (18.25)$$

We complete the boundary conditions as usually in the case of free source jets, by the integral conditions of conservation

$$\int_{-\infty}^{+\infty} (\rho u^2 - \mu_m H^2)\, dy = \text{const}, \qquad (18.26)$$

which is obtained by integrating the first equation of System (18.13) across the boundary layer, using the condition of zero divergence of the vectors v and H. Taking the equality $\alpha = \omega$ derived above into account, we obtain from this

$$\alpha = \omega = -\frac{1}{3}, \quad \beta = -\frac{2}{3}.$$

As we see from the latter equations, the values of the constants of self-similarity, α and β, are the same as in the case of the laminar plane source jet of nonconducting fluid.

The constants A, B and H are determined in the following; for this purpose we make use of the fact that Integral (18.26) for the self-similar problem can be written in the form

$$\frac{\rho A^2}{B}\int_{-\infty}^{+\infty} F'^2\,d\varphi - \frac{\mu_m H^2}{B}\int_{-\infty}^{+\infty} h'^2\,d\varphi = \text{const}, \tag{18.27}$$

from which it follows that each of the integrals in Eq. (18.26) represents a constant quantity. In other words, in the problem considered, the total flux of mechanical momentum and the total flux of electromagnetic momentum are conserved separately.

We use the denotations

$$J_x = \frac{\rho A^2}{B}\int_{-\infty}^{+\infty} F'^2\,d\varphi = \text{const}, \quad J_{mx} = \frac{\mu_m H^2}{B}\int_{-\infty}^{+\infty} h'^2\,d\varphi = \text{const}. \tag{18.28}$$

The problem is thus reduced to an integration of the system of equations

$$\left.\begin{aligned} F''' + (2FF')' - 2Shh'' = 0,\\ h'' + 2\mathrm{Pr}_m(Fh' - F'h) = 0,\end{aligned}\right\} \tag{18.29}$$

which are obtained instead of (18.16) and (18.17) if we use the values given above for the constants of self-similarity and if we put

$$\frac{A}{B^2} = 6\nu. \tag{18.30}$$

As to its physical meaning the parameter $S = \frac{\mu_m H^2}{\rho A^2} \sim \frac{J_{mx}}{J_x}$, entering the first of Eq. (18.29) is a measure of the ratio between magnetic momentum flux and mechanical momentum flux or the ratio of total magnetic energy to total kinetic energy. It may also be expressed as the ratio between the conventional initial values of the square velocity of the Alfvén waves, $V_{Am}^2 = \frac{\mu_m H_m^2}{\rho}$ and the square velocity u_m^2 of motion along the axis of the jet.

The ratio $S = \frac{V_{Am}^2}{u_m^2}$ differs only by a numerical factor from the ratio of the squares of the conventional initial values of V_{A0}^2 and u_0^2 when the latter is determined by the following equation:

$$J_{mx} = \int_{-\infty}^{+\infty} \mu_m H^2 \, dy = \mu_m H_0^2 b_0, \quad J_x = \int_{-\infty}^{+\infty} \rho u^2 \, dy = \rho u_0^2 b_0.$$

In this case

$$S = \frac{V_{Am}^2}{u_m^2} = \frac{\int_{-\infty}^{+\infty} F'^2(\varphi) \, d\varphi}{\int_{-\infty}^{+\infty} h'^2(\varphi) \, d\varphi} \frac{V_{A0}^2}{u_0^2}.$$

In the case of similar distributions with $F'(\varphi) = h'(\varphi)$ we also have $S = S_0 = \frac{V_{A0}^2}{u_0^2}$; this case ($\sigma = \infty$) will be considered below.

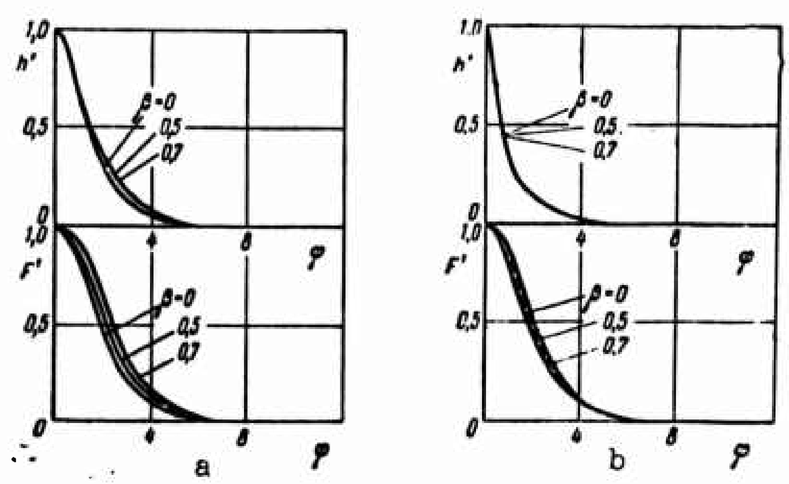

Fig. 18.4. Distributions of velocity and longitudinal magnetic field component. a) $Pr_m = 1$; b) $Pr_m = 4$ [315].

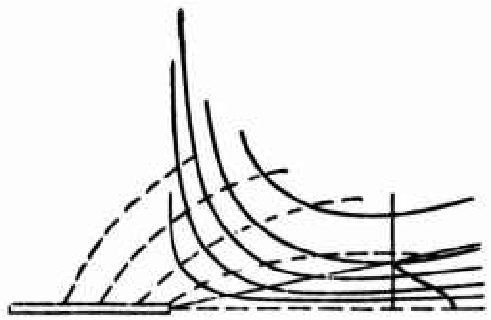

Fig. 18.5. Diagram of the streamlines and the magnetic lines of force in a jet.

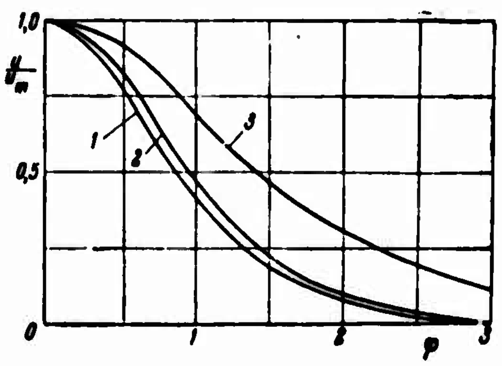

Fig. 18.6. Velocity distributions in the jet. 1. $S = 0$; 2. $S = 0{,}4$; $Pr_m = 1$; 3. $S = 0{,}4$; $Pr_m = \infty$.

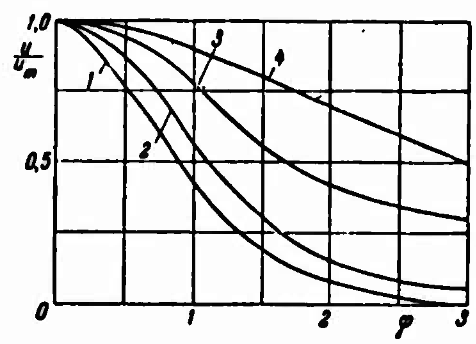

Fig. 18.7. Distributions of longitudinal velocity component for various values of the parameter S ($Pr_m = \infty$). 1. $S = 0$; 2. $S = 0{,}2$ 3. $S = 0{,}5$; 4. $S = 0{,}7$.

It should be noted – and this is essential – that a solution to the problem considered will only exist if $0 \leqslant S \leqslant 1$. The lower limit, $S = 0$ corresponds to a jet of a nonconducting fluid, the upper limit, $S = 1$, to the limiting value of magnetic momentum.

From Condition (18.28) and Eq. (18.30) we obtain the values of the constants

$$A=\left[\frac{J_x}{\sqrt{6\nu\rho}}\left(\int\limits_{-\infty}^{+\infty}F'^2\,d\varphi\right)^{-1}\right]^{2/3},$$

$$B=\sqrt[3]{\frac{A}{6\nu}},$$

$$H=\sqrt{\frac{S}{\mu_m}}\left(\int\limits_{-\infty}^{+\infty}h'^2\,d\varphi\right)^{-1/2}\times$$

$$\times\left(\frac{J_x}{6\nu}\right)^{1/3}\left(\rho\int\limits_{-\infty}^{+\infty}F'^2\,d\varphi\right)^{-1/6}.$$

The results of a numerical integration of the system of equations (18.29) with the boundary conditions (18.25) are contained in the papers [315, 316].

Let us here give some of them.

Figure 18.4 shows the curves of the relative velocity $\frac{u}{u_m}=F'(\varphi)$ and the magnetic field component $\frac{H_x}{H_{xm}}=h'(\varphi)$ for various values of the parameter S with $Pr_m = 1$ and $Pr_m = 4$.

To render it clearer, we show in Fig. 18.5 a schematic diagram of the streamlines and the magnetic lines of force. From the qualitative point of view, these graphs verify the solution of L.D. Landau's generalized problem.

In Fig. 18.6 we compared the relative velocity distributions for $Pr_m = 1$ and for a perfectly conductive fluid. This comparison shows that the numerical solution [315, 316] deviates only insignificantly from the analytical solution for infinite conductivity. The influence of the parameter S on the distribution curve of the longitudinal velocity component is shown in Fig. 18.7 for the example of $\sigma = \infty$.

18.5. THE MAGNETIC BOUNDARY LAYER OF SECOND KIND

To illustrate the solution of the equations of the magnetic boundary layer of second kind, we consider by way of example the problem of the expansion of a plane laminar semilimited source jet of a conducting

fluid [89,90].

The initial self-similar equations of the problem are Eqs. (18.19) and (18.20) completed by the energy equations

$$\left.\begin{aligned} &F''' + \frac{A}{2\nu B^3}[(\alpha+1)FF'' - 2\alpha F'^2] = 0, \\ &h'' + \frac{A}{2\nu B^3}\mathrm{Pr}_m[(\alpha+1)Fh' - 2\omega F'h] = 0, \\ &\theta'' + \mathrm{Pr}\,F\theta' + \frac{4\nu_m}{\rho c_p(T_w - T_\infty)}\frac{H_w^2 B^3}{A}(h')^2 = 0. \end{aligned}\right\} \tag{18.31}$$

(Here $\theta = \frac{T-T_\infty}{T_w-T_\infty}$, T_w is the constant value of the plate surface temperature $H = H_w h(\varphi)$, H_w is the constant value of the magnetic field strength on the surface of the plate.)

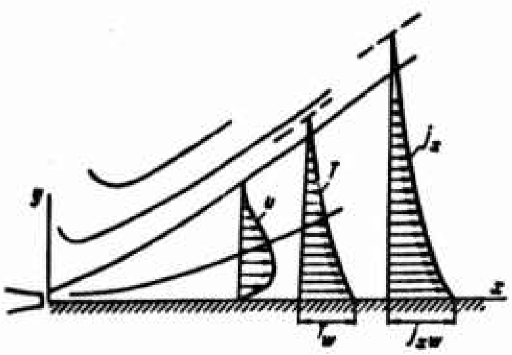

Fig. 18.8. Schematic representation of semilimited jet of conducting fluid

The dynamic problem, which is represented by the first equation of System (18.31), with the corresponding boundary conditions $F = F' = 0$ with $\varphi = 0$, $F' = F'' = 0$ with $\varphi = \infty$, and the integral invariant

$$\int_0^\infty \rho u^{3/2}\,dy = \text{const},$$

is solved independently from the others. When deriving its solution, which was given previously, we find the constants of self-similarity, $\alpha = -1/2$, $\beta = -3/4$, and the constants A and B (see Table 7.1) used in the integration of the other two equations of System (18.31).

We choose the boundary conditions for the problem on the basis of the following physical statement (see diagram in Fig. 18.8).

Let us assume that an electric current flows in the direction of the Ox axis through the plate whose outer surface is insulated against the moving fluid (such that the electrical contacts can only be at the front and rear edges of the plate). The magnetic field induced by the

current is directed along the Oz axis.

The boundary conditions for the magnetic field and the temperature therefore have the form

$$h=1,\ \theta=1 \text{ with } \varphi=0, \qquad h=0,\ \theta=0 \text{ with } \varphi=\infty. \tag{18.32}$$

The solution of the induction equation

$$h''+\mathrm{Pr}_m F h'=0, \tag{18.33}$$

obtained by transforming the second equation of System (18.31) has the form

$$h(\varphi)=1-\int_0^{\varphi}\exp\left(-\mathrm{Pr}_m\int_0^z F\,dz\right)dz\left[\int_0^{\infty}\exp\left(-\mathrm{Pr}_m\int_0^z F\,dz\right)dz\right]^{-1}. \tag{18.34}$$

As regards the nonuniformity of the heat propagation equation obtained from the last equation of System (18.31)

$$\theta''+\mathrm{Pr}\,F\theta'+4\frac{\mathrm{Pr}}{\mathrm{Pr}_m}\frac{H_w^2}{\rho c_p(T_w-T_\infty)}(h')^2=0, \tag{18.35}$$

its general solution is represented in the form

$$\theta(\varphi)=1-\left[\int_0^{\varphi}\exp\left(-\mathrm{Pr}\int_0^z F\,dz\right)dz\right]\left[\int_0^{\infty}\exp\left(-\mathrm{Pr}\int_0^z F\,dz\right)dz\right]^{-1}- -4\frac{\mathrm{Pr}}{\mathrm{Pr}_m}\frac{H_w^2}{\rho c_p(T_w-T_\infty)}\int_0^{\varphi}(h')^{\frac{\mathrm{Pr}}{\mathrm{Pr}_m}}\left[\int_0^z (h')^{2-\frac{\mathrm{Pr}}{\mathrm{Pr}_m}}dz\right]dz. \tag{18.35'}$$

In this way the problem is solved.

The solution of the induction equation (18.33) in the form of (18.34) corresponds to the above-mentioned analogy with the thermal problem in a nonconducting fluid (neglecting the heat of friction). The solution of the energy equation (18.35') consists of two parts. The first of them, which contains two terms, is independent of the presence of a magnetic field; the second is determined by Joulean dissipation whose action on the temperature distribution is analogous to that of the heat of friction.

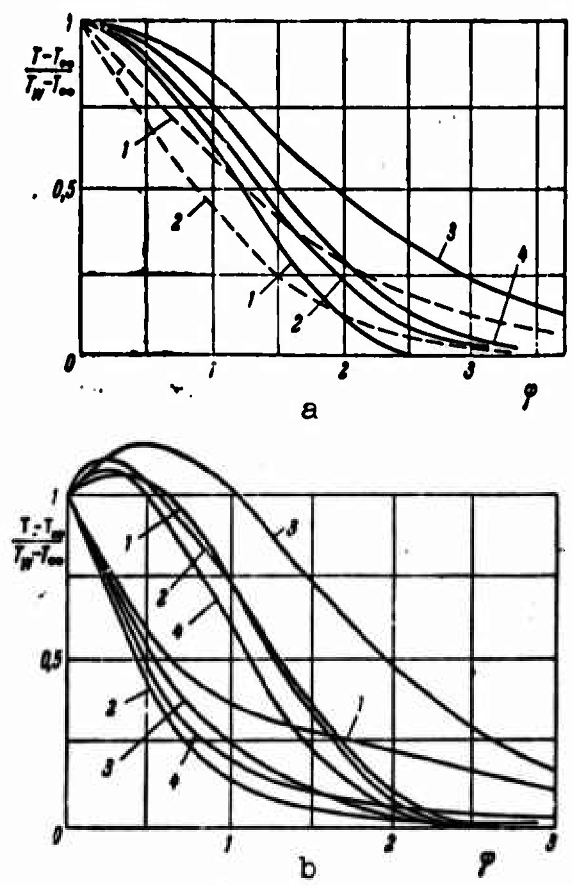

Fig. 18.9. Influence of the magnetic field on the temperature distribution in a semilimited jet

[Т°Г. a) ——— $Y = 1$; — — — $Y = 0$; б) ——— $Y = 2$, — — — $Y = -1$ [$Y = H_w^2/\rho c_p (T_w - T_\infty)$]. 1. $Pr = 0,5$; $Pr_m = 1,0$; 2. $Pr = 1,0$; $Pr_m = 0,5$; 3. $Pr = 0,5$; $Pr_m = 0,5$; 4. $Pr = 1,0$; $Pr_m = 1,0$.

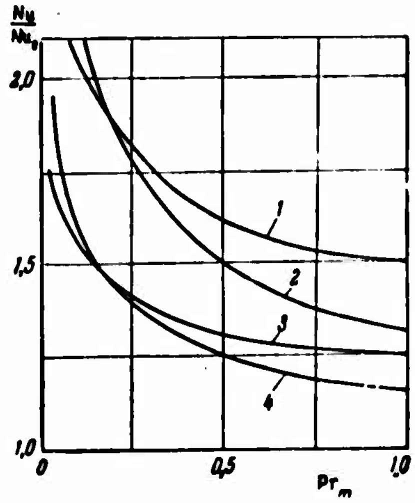

Fig. 18.10. Influence of a magnetic field on the heat transfer [90]. 1. $N_m = 1$; $Pr = 1$; 2. $N_m = 1$; $Pr = 0.5$; 3. $N_m = 0.5$; $Pr = 1$; 4. $N_m = 0.5$; $Pr = 0.5$.

The graphs in Figs. 18.9 and 18.10 illustrate the influence of the magnetic field on the temperature distribution in the semilimited jet, and on the heat transfer [89,90]. The first of these figures needs no explanation. From the second we see that in the problem considered the heat transfer is the more intense the higher the energy of the magnetic field but, compared with the heat transfer in the case of a nonconducting fluid, the increase in heat transfer is the smaller the larger the value of the magnetic Prandtl number Pr_m.

Results analogous in nature may also be obtained in other cases of self-similar jet flows applicable to the theory of the magnetic boundary layer of second kind.

18.6. JETS WITH SMALL VALUES OF THE MAGNETIC REYNOLDS NUMBER

In the case of very small values of the magnetic Reynolds number $Re_m \ll 1$ in problems of applied magnetohydrodynamics an approximation

method of solution has become widespread, in which the induction equation is not taken into consideration. In this case, it is only the usual boundary layer equations which are integrated, i.e., the equations of continuity, motion and energy, where two of them are completed by terms describing the influence of the magnetic field on the flow of a conducting viscous fluid.

With this statement of the problem where the magnetic field is given, with respect to both magnitude and direction, we neglect the induced magnetic field. It is natural in this case to consider the jet flow of a conducting fluid in a direction normal to the magnetic field applied. In this case, the interaction between fluid and field is obviously maximum.

In connection with this, let us turn to the problem of expansion of a plane laminar source jet in a transverse magnetic field [91] which was considered for the first time by H. Jungklaus [253] (see also [274]).

The equations of motion and continuity and the equation describing the propagation of heat are written in the following form:

$$\left.\begin{aligned} &u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}-\frac{\sigma\mu_m^2}{\rho}H^2u, \quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\\ &u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=a\frac{\partial^2 T}{\partial y^2}+\frac{\sigma\mu_m^2}{\rho c_p}H^2u^2. \end{aligned}\right\} \qquad (18.36)$$

In the first of these equations, the second term of the righthand side represents the electromagnetic volume force per unit mass

$$F_{\text{эм}}=\frac{\mu_m(j\times H)_x}{\rho}, \text{ where } H(0, H_y, 0), \; j(0, 0, j_z),$$

where $j=\sigma\mu_m(u\times H)$ is the electric current density.

The new term in the righthand side of the last equation (new as compared to the usual problem) represents the separation of Joulean heat per unit mass:

$$q_{\text{дж}}=\frac{j^2}{\rho\sigma}.$$

The boundary conditions for the velocity in Eq. (18.36) is maintained in the same formulas in the case of the jet of nonconducting fluid:

$$\left.\begin{array}{ll} v=0, & \frac{\partial u}{\partial y}=0 \text{ with } y=0, \\ u=0 & \text{ with } y=\pm\infty. \end{array}\right\} \qquad (18.37)$$

Just as in other cases of source jets, the solution of the problem will be sought for the self-similar flow. Besides the ordinary transformation formulas for the velocity of motion and the coordinates

$$\frac{u}{u_m}=F'(\varphi), \quad u_m=Ax^{\alpha}, \quad \varphi=Byx^{\beta} \qquad (18.38)$$

we also use a power function to describe the dependence of the applied magnetic field on the coordinate x:

$$H=H_0x^{\omega}, \quad H_0=\text{const}. \qquad (18.39)$$

Let us first consider the dynamic problem. As regards the thermal problem, it is, as in the other cases, more convenient to consider it thereafter, taking into account the different boundary conditions for the temperature.

The equations of motion and continuity are transformed with the help of the above self-similarity transformation formulas to an ordinary differential equation

$$F'''+3(\alpha+1)FF''-6\alpha F'^2-NF'=0 \qquad (18.40)$$

with the boundary conditions

$$F'(0)=1, \quad F(0)=F'(\infty)=0 \qquad (18.41)$$

In the derivation of Eq. (18.40), in order to make it independent of the coordinate x, we assumed $\beta=\frac{\alpha-1}{2}$ and, moreover, $\omega=\beta$, as this is usual for plane laminar source jets. In addition to this, for the relation between A and B (as one of them is arbitrary) we assumed $A/B^2 = 6\nu$ and, finally, we used the denotation $N=\frac{6\sigma\mu_m^2H_0^2}{A\rho}$. As to its physical meaning the parameter N characterizes the magnetic effect on the motion

(note that always $N > 0$).

In order to determine the value of the constant α entering the expression for the law of velocity decrease on the jet axis $u_m = Ax^{\alpha}$, we integrate Eq. (18.40) across the jet, between the limits $\varphi = 0$ and $\varphi = \infty$. Here we put

$$\alpha = -\frac{1}{3}\left(1 + \frac{N}{3}\frac{F(\infty)}{\int\limits_0^{\infty} F'^2 d\varphi}\right). \quad (18.42)$$

After the substitution of the expression obtained for the constant α (which here depends on the magnetic parameter N, which is specific for the problem considered on the jet in a magnetic field) in Eq. (18.40), the latter can be rewritten in the form

$$F''' + 2(FF'' + F'^2) - \frac{N}{3}\frac{F(\infty)}{\int\limits_0^{\infty} F'^2 d\varphi}\left[FF'' - 2F'^2 + 3\frac{\int\limits_0^{\infty} F'^2 d\varphi}{F(\infty)} F'\right] = 0. \quad (18.43)$$

This equation is a generalization of the equation for a plane source jet of nonconducting fluid. In fact, with $N = 0$, Eq. (18.43) passes over to the equation

$$F''' + 2(FF'' + F'^2) = 0, \quad (4.47)$$

whose solution under the same boundary conditions was considered in Section 4.3.

It is easy to prove (e.g., by substituting) that the solution of this equation (for the case of $N = 0$)

$$F'(\varphi) = (\mathrm{ch}\,\varphi)^{-2} \quad (18.44)$$

is also a solution to the generalized equation (18.43).

Using here Solution (18.44), we obtain from Eq. (18.42) the final expression for the constant α as a function of the magnetic parameter N:

$$\alpha = -\frac{1}{3}\left(1 + \frac{N}{2}\right). \quad (18.45)$$

We also write the constant β (and the constant ω is equal to it) in terms of the parameter N:

$$\beta = \omega = \frac{\alpha - 1}{2} = -\frac{8+N}{2}. \qquad (18.46)$$

In order finally to solve the problem we must also determine the values of the constants A and B (the quantity H_0 characterizing the applied magnetic field is given). For this purpose, just as with other problems, we have recourse to an integral condition. As already mentioned, the exponential self-similarity transformation formulas (18.38) predetermine the validity of an equation of the form

$$\overline{K} = \int_0^\infty u^{\beta/\alpha}\,dy = \text{const}, \qquad (18.47)$$

where $\frac{\beta}{\alpha} = \frac{8+N}{2(2+N)}$. With $N = 0$ Eq. (18.47) passes over to the condition of momentum flux conservation in the projection on the x axis. Let us reckon the constant $\overline{K}$ defined by the above equation (18.47) to the given parameters of the problem*. In this case the solution of the problem will be unambiguous and all constants needed are determined unambiguously as functions of the magnetic parameter N. From Eq. (18.47) we obtain

$$A = \left[\overline{K}B\left(\int_0^\infty (F')^{\frac{8+N}{2(2+N)}}\,d\varphi\right)^{-1}\right]^{-\frac{2}{8+N}}, \quad \frac{A}{B^2} = 6\nu. \qquad (18.48)$$

Let us also formulate expressions for the rate of fluid flow through the jet cross section, the momentum flux and the flux of kinetic energy in terms of the coordinate x and N:

$$\left.\begin{aligned} G &= \int_0^\infty \rho u\,dy = \frac{A}{B}\rho\int_0^\infty F'(\varphi)\,d\varphi\cdot x^{\alpha-\beta} \sim x^{\frac{4-N}{12}}, \\ J_x &= \int_0^\infty \rho u^2\,dy \sim x^{2\alpha-\beta} \sim x^{-\frac{N}{4}}, \\ E &= \int_0^\infty \rho u^3\,dy \sim x^{3\alpha-\beta} \sim x^{-\frac{4+3N}{12}}. \end{aligned}\right\} \qquad (18.49)$$

The whole problem has thus been led to a final solution. In this solution to each value chosen for the parameter N a certain definite value of the constants α, β, A, B, etc. corresponds.

As already indicated, the value $N = 0$ corresponds to the case of a jet of nonconducting fluid (or the absence of an applied magnetic field). The range of the physically reasonable values of N is therefore limited by the minimum value of $N_{min} = 0$. Physical considerations, already mentioned above, also permit the establishment of the upper limit, i.e., the maximum value of the parameter N, such that from the solution the domain can be separated which corresponds to the physical problem on the motion of a source jet in a transverse magnetic field. On the assumption that the fluid flow rate in the jet must not decrease we obtain $N_{max} = 4$ from the first formula of (18.49) and, consequently, $0 < N < 4$.

In the table contained in the test we have compiled the values of the characteristic constants of the problem, which correspond to different values of the magnetic parameter N within the limits given.

N	$\beta = \omega$	α	β/α	γ
0	$-2/3$	$-1/3$	2	$-1/3$
1	$-3/4$	$-1/2$	$3/2$	$-1/4$
2	$-5/6$	$-2/3$	$5/4$	$-1/6$
3	$-11/12$	$-5/6$	$11/10$	$-1/12$
4	-1	-1	1	0

In the last column of this table we also find the values of the constant of self-similarity, γ, entering the expression for the surplus temperature on the jet axis: $\Delta T_m = \Gamma x^{\gamma}$. These values of γ are obtained from the solution of the thermal problem for two variants of boundary conditions for the temperature: symmetrical, with velocity conditions (see Table case a) and asymmetric (for which $\gamma = 0$ case b).

Let us briefly consider this solution which applies to the self-

similar approximation for a flow far away from the source. Returning to the third equation of (18.36) we note that the term entering it, which describes Joulean heat, decreases more rapidly than the others as the distance from the source increases, as may be seen from the solution. For the range of the self-similar solution it can therefore be neglected. With this simplification, the problem can be solved in the self-similar approximation, also with respect to the temperature distribution.

We denote as usually

$$\Delta T = \Delta T_m \theta(\varphi), \quad \Delta T_m = \Gamma x^{\gamma}. \qquad (18.50)$$

The two variants of the temperature boundary conditions are the following: case (a):

$$\left.\begin{aligned} \theta &= 1 \text{ with } \varphi = 0, \\ \theta &= 0 \text{ with } \varphi = \pm\infty; \end{aligned}\right\} \qquad (18.51)$$

case (b):

$$\left.\begin{aligned} \theta &= 1 \text{ with } \varphi = +\infty, \\ \theta &= 0 \text{ with } \varphi = -\infty. \end{aligned}\right\} \qquad (18.52)$$

Let us first solve the problem for case (a) with the boundary conditions (18.51) which are symmetrical with respect to both velocity and temperature.

The self-similar equation of heat propagation in this case reads

$$\theta'' + 3\mathrm{Pr}\,[(1+\alpha)F\theta' - 2\gamma F'\theta] = 0. \qquad (18.53)$$

In contrast to the dynamic problem, the thermal problem in the statement considered has a general integral invariant, namely the condition of conservation of the flux of surplus heat content:

$$Q = \int_{-\infty}^{+\infty} \rho c_p u \Delta T dy = \text{const}. \qquad (18.54)$$

In agreement with it we obtain

$$\gamma = -\frac{\alpha+1}{2} = -\frac{1}{3}\left(1-\frac{N}{4}\right), \quad \Gamma = \frac{BQ}{\rho c_p A}\left(\int_{-\infty}^{+\infty} F'\theta\, d\varphi\right)^{-1}. \qquad (18.55)$$

With these conditions the solution of Eq. (18.53), taking the boundary conditions into account, assumes the form

$$\theta(\varphi)=[F'(\varphi)]^{\frac{3}{2}(1+\alpha)Pr}=(\operatorname{ch}\varphi)^{-2\left(1-\frac{N}{4}\right)Pr}. \qquad (18.56)$$

It is interesting to remark that, unlike the usual ("nonmagnetic") case of jet, in the problem considered with a Prandtl number equal to unity, the velocity and temperature distributions will not be similar. It is obvious that the distributions of u and ΔT will only be similar if the value of $Pr=(1-\frac{N}{4})^{-1}$.

For the second case of boundary conditions it follows from physical considerations that, as in the common jet, we must assume $\gamma=0$ and $\Gamma=T_1-T_2=\mathrm{const}$.

From the equation

$$\theta''+3(\alpha+1)Pr(F\theta)'=0 \qquad (18.57)$$

with the boundary conditions (18.52) we obtain

$$\theta(\varphi)=\left[\int_{-\infty}^{+\infty}(\operatorname{ch}\varphi)^{-2\left(1-\frac{N}{4}\right)Pr}d\varphi\right]^{-1}\int_{-\infty}^{\varphi}(\operatorname{ch}\varphi)^{-2\left(1-\frac{N}{4}\right)Pr}d\varphi. \qquad (18.58)$$

In this case the flux of surplus heat content is proportional to the flow rate: $Q\sim G\sim x^{\alpha-\beta}\sim x^{\frac{4-N}{12}}$.

The solution obtained for the problem of the expansion of a plane source jet of conducting fluid in a transverse magnetic field with small values of the magnetic Prandtl number refers to laminar conditions of motion. The influence of the field on the laws of jet expansion resulted chiefly in an alteration of the law of velocity decrease along the jet axis with increasing nozzle distance. The electromagnetic volume force, which arises on the intersection of the lines of force of the magnetic field by the conducting medium, it directed against the initial momentum of the jet thus decelerating the motion. With low values of the magnetic Reynolds number $(Re_m \ll 1)$ and neglecting the influence of

the induced magnetic field, we may, in a first approximation, extend the self-similar solution obtained to the turbulent motion of fluid in a jet. Here, of course, the question of the value of the empirical constant a entering the expression for the reduced coordinate φ (and through it also the constant A, etc.) is still open. More than that, it can only be shown by experiment whether it is possible with the help of a single empirical constant to account for not only the decelerating action of the magnetic field on the flow, but also another characteristic field effect, namely the quenching of the turbulent pulsations [121,123]. Without entering into details with respect to this problem, we restrict ourselves to the same approximation as used in the theory of self-similar turbulent jets of incompressible fluid (Section 11.2). In other words, as we want to obtain at least an approximate image of the expansion of self-similar turbulent jets of conducting fluid, we assume as in the "nonmagnetic" case

$$\nu_T = bu_m \sim x^{\alpha-\beta} \sim x^{\alpha+1} \, (\beta = -1). \qquad (18.59)$$

Under this supposition, we can again use the advantage of solving the problem by the method of the asymptotic boundary layer such that the nondimensional equations of self-similar jets and their solutions remain unchanged, applying to both laminar and turbulent motions. The values of the constants of self-similarity, the laws of velocity variation along the jet axis, of the jet's effective boundary, etc. are, of course, different in these two cases. In particular, for turbulent, self-similar jets we must assume $\beta = -1$ and correspondingly, $\varphi = \frac{y}{ax} \equiv Byx^{-1}$ as the effective boundary of the turbulent source jet is a straight line, just as in the "nonmagnetic" case. The constants α and ω will depend on the value of the magnetic parameter N as in the laminar jet. Determining the relation between them in the same way as above, in the case of the laminar jet, we obtain for the turbulent jet

$$\alpha = -\frac{2+3N}{4}, \quad \omega = -\frac{6+3N}{8}. \qquad (18.60)$$

In this case the integral equation has the form

$$\overline{K} = \int_0^\infty u^{-1/\alpha}\, dy = \text{const},$$

and consequently

$$A = \left[B\overline{K}\left(\int_0^\infty (F')^{\frac{4}{2+3N}}\, d\varphi\right)^{-1}\right]^{\frac{2+3N}{4}}.$$

Also calculating the coordinate dependence of flow rate, momentum flux and kinetic energy flux

$$G \sim x^{\frac{2-3N}{4}}, \quad J_x \sim x^{\frac{0-6N}{4}}, \quad E \sim x^{\frac{10-3N}{4}},$$

we obtain from the first of these expressions the range of values of the magnetic parameter in which the fluid flow rate will not decrease:

$$0 < N < \frac{2}{3}. \qquad (18.61)$$

As already shown in the discussion of the results for the laminar jet, this limitation corresponds to the physical statement of the problem on a source jet of conducting fluid expanding in a transverse magnetic field. In particular, $N = 0$ corresponds to an ordinary jet.

As regards the form of the velocity distribution curve of a turbulent jet, as already mentioned, its expression in nondimensional form remains quite the same as with the laminar jet (or with laminar and turbulent "nonmagnetic" self-similar jets):

$$\frac{u}{u_m} = F'(\varphi) = \mathrm{ch}^{-2}\varphi. \qquad (18.62)$$

We can generalize analogously the solution derived above for the thermal problem of a laminar jet to the thermal problem of the turbulent jet of a conducting fluid with $\mathrm{Re}_m \ll 1$ and the corresponding boundary

conditions. Such a generalization, which has not been given here for the sake of brevity, may play the part of an approximate image of the temperature distribution, just as in the dynamic problem. A more complete judgment on the admissibility of the approximate expression $v_T = b_T u_m$, and on the values of the empirical constant $a \equiv \frac{1}{B}$ in the presence of a magnetic field, etc. can only be obtained as the result of experiments.

It must be finally noted that the method of solving self-similar problems on the expansion of jets of conducting fluids in a transverse magnetic field with small values of the magnetic Reynolds number may be also applied analogously to other problems, for example on the semi-limited jet (plane and fan type), on the free fan type jet and others. It is a general feature of these solutions that the self-similar solution of the problem does not exist for all values of the magnetic parameter and, as in the examples considered, and to each value of the magnetic parameter (within these limits) there exists its solution, its values of the constants α, ω, etc.

In this connection it is expedient in the last Section of this Chapter to consider the solution to this problem on the plane source jet in the transverse magnetic field with $Re_m \ll 1$, without any suppositions on self-similarity. Note however that, from the physical point of view, the results do not differ from those obtained above.

18.7. PLANE JET IN UNIFORM MAGNETIC FIELD

For comparison with the results obtained in the preceding section, we again consider with small values of the magnetic Reynolds number $Re_m \ll 1$ the problem on the expansion of a plane source jet in a given uniform transverse magnetic field. Just as previously we shall not take the induction equation into account. The motion in the jet will first be considered laminar and then (under the same suppositions as above,

$v_T = bu_m$) we generalize the results obtained to turbulent motion. Since the assumption on the uniformity of the magnetic field applied is contradictory to the self-similarity of the flow (in the solutions given above the constant ω vanished nowhere), the final solution of the problem is connected with numerical calculations and its expansion in a series with respect to a small parameter. This solution (and some numerical illustrations of it) was obtained in R. Piskin's paper [275]; we shall report briefly its contents in the following (using the denotations chosen in this book).

The equations of motion and continuity are represented in the form

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma\mu_m^2 H_0^2}{\rho}u, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{18.63}$$

with the boundary conditions for the plane source jet

$$\left.\begin{array}{ll} v = 0, \ \frac{\partial u}{\partial y} = 0 & \text{with } y = 0, \\ u = 0 & \text{with } y = \pm\infty. \end{array}\right\} \tag{18.64}$$

The magnitude of the magnetic field $H_0 = H_{0y}$ is assumed constant. As the parameter of the problem we choose the quantity

$$\overline{N} = \frac{\sigma\mu_m^2 H_0^2}{\rho u_{m0}} = \frac{\sigma\mu_m^2 H_0^2}{\rho A} x^{1/3} = N x^{1/3}, \tag{18.65}$$

where $u_{m0} = Ax^{-1/3}$ is the maximum velocity (on the axis) of a nonmagnetic free plane laminar jet (with $H_0 = 0$ or $\sigma = 0$). Note that, according to the physical meaning, the parameter introduced represents the ratio between the square Hartmann number

$$\mathrm{Ha} = \mu_m H_0 x\sqrt{\frac{\sigma}{\nu\rho}}$$

and the hydrodynamic Reynolds number

$$\mathrm{Re}_x = \frac{u_m x}{\nu},$$

determined from the velocity u_{m0} in the case of $H_0 = 0$. The quantities referring to a jet of nonconducting fluid are denoted by the subscript

"0". For this case, we obtain from the above relations

$$u = Ax^{-1/3}F'(\varphi), \quad \varphi = Byx^{-2/3}.$$

The values of the constants A and B were obtained in Section 4.3 in the form

$$A = \frac{1}{2}\left(\frac{3J_x^2}{4\rho^2\nu}\right)^{1/3}, \quad B = \frac{1}{2}\left(\frac{J_x}{6\rho\nu^2}\right)^{1/3}.$$

We represent the streamfunction for the flow with $H_0 \neq 0$ in the form of a power series of the parameter $\overline{N} = N \cdot x^{4/3}$:

$$\psi = \frac{A}{B}x^{1/3} \sum_{n=0}^{\infty} (\overline{N})^n F_n(\varphi). \tag{18.66}$$

According to the usual formulas $\left(u = \frac{\partial\psi}{\partial y},\ v = -\frac{\partial\psi}{\partial x}\right)$ we then have for the velocity components in the jet of conducting fluid

$$u = Ax^{-1/3} \sum_{n=0}^{\infty} (Nx^{4/3})^n F_n'(\varphi), \tag{18.67}$$

$$v = -\frac{A}{3B}x^{-2/3}\left[\sum_{n=0}^{\infty} (\overline{N})^n 4nF_n(\varphi) + \right.$$
$$\left. + \sum_{n=0}^{\infty} (\overline{N})^n F_n(\varphi) - \sum_{n=0}^{\infty} (\overline{N})^n 2\varphi F_n'(\varphi)\right]. \tag{18.68}$$

We substitute these expressions in the initial equation of motion and equate the coefficients of equal powers of the parameter $\overline{N}$ (or, what is the same, the coefficients of the same powers of x). As the result we obtain a system of ordinary linear differential equations for the determination of the function $F_n(\varphi)$.

For the zero approximation

$$F_0''' + 2(F_0F_0'' + F_0'^2) = 0. \tag{18.69}$$

For the first approximation

$$F_1''' + 2F_0F_1'' - 8F_0'F_1' + 8F_0''F_1 + 4F_0'F_1 + F_0''F_1 - 6F_0' = 0 \tag{18.70}$$

etc. (The higher approximations are so cumbersome that we do not give them.) The boundary conditions corresponding to these equations read

$$F_n = F_n' = 0 \text{ with } \varphi = 0, \quad F_n' = 0 \text{ with } \varphi = \pm\infty. \qquad (18.71)$$

The zero approximation equation obviously corresponds to a jet of nonconducting fluid for which

$$\frac{u}{u_m} = F_0'(\varphi), \quad F_0' = \mathrm{ch}^{-2}\varphi.$$

The solution in first approximation, in agreement with the expansion used, has the form

$$\frac{u}{u_m} = \mathrm{ch}^{-2}\varphi + Nx^{4/3}F_1'(\varphi),$$

where $F_1'(\varphi)$ is a function determined by numerical integration of the first approximation equation (18.70). The values of $F_1'(\varphi)$, corresponding to the series of values of the coordinate φ in the interval between $\varphi = 0$ on the jet axis to $\varphi = 3.4$ (which corresponds approximately to the effective boundary of the jet), have been taken from paper [275] and are compiled in the following table:

φ	$F_1'(\varphi)$	φ	$F_1'(\varphi)$	φ	$F_1'(\varphi)$
0,0	−0,7500	1,2	+0,0001	2,4	+0,0331
0,2	−0,6023	1,4	+0,0388	2,6	+0,0258
0,4	−0,5442	1,6	+0,0535	2,8	+0,0195
0,6	−0,3617	1,8	+0,0547	3,0	+0,0108
0,8	−0,1965	2,0	+0,0492	3,2	+0,0080
1,0	−0,0751	2,2	+0,0412	3,4	+0,0058

Let us give some illustrations. Figure 18.11 shows the distributions of the longitudinal velocity component which are drawn for the first approximation with three values of the parameter $\bar{N} = 0$, 0.2 and 0.5. The first of them corresponds to a jet of nonconducting fluid; the velocity values u_{m0} on the jet axis corresponding to it are used as a scale for all curves in Fig. 18.11. We see from the figure that the velocity on the jet axis in the whole cross section decreases more rapidly as the magnetic parameter N increases (with one and the same

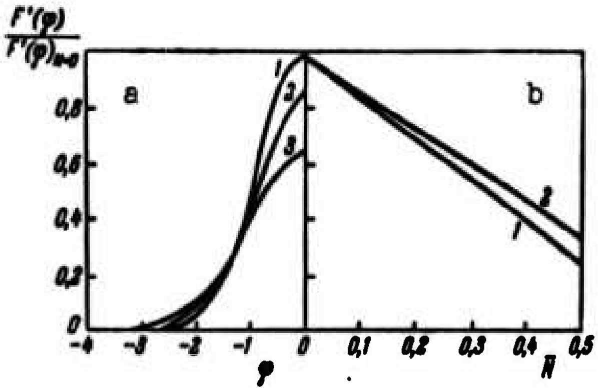

Fig. 18.11. Magnetic field effect on velocity distribution in plane jet [275].
а) 1. $\bar{N} = 0$; 2. $\bar{N} = 0,2$; 3. $\bar{N} = 0,5$; б)
1) Laminar; 2) turbulent jet.

value of x). The value of the transverse velocity component at infinity also decreases; it determines the injection of liquid in the flow and, in connection with this, also the flow rate in the jet with N 0 relative to the jet of nonconducting fluid. In a first approximation these quantities (v_∞ and G) can be estimated from the expressions

$$\frac{v_\infty}{v_{\infty 0}} \approx 1 - \frac{4}{3} N x^{4/3}, \quad \frac{G}{G_0} \approx 1 - \frac{3}{4} N x^{4/3}. \qquad (18.72)$$

We see from the latter formula that for the first approximation the value of the parameter $\bar{N}$ characterizing the increase in rate of flow must be within the limits $0 < \bar{N} < 4/3$. From the qualitative point of view all these results agree completely with those given above for the self-similar laminar jet of conducting fluid.

Let us now turn to the case of a turbulent jet, assuming that, as mentioned already, the coefficient of turbulent kinematic viscosity is given by the same formula as in the case of the self similar jet of nonconducting fluid: $v_T = b \cdot u_m$. We also assume for the source jet (i.e., the flow far away from the nozzle, though it is not self-similar owing

to the uniformity of magnetic field) for ghe generalized coordinate $\varphi = Byx^{-1}$, i.e., $\beta = -1$.

The expansion parameter for the turbulent jet will depend on the x coordinate according to the law $\bar{N}_T = Nx^{2/3}$, where, as previously $N = \sigma\mu_m^2 H_0^2/\rho A$, whereas $u_{m0} = Ax^{-1/3}$.

Having recourse to an analogous expansion of the streamfunction and the velocity components in series of the small parameter $\bar{N}_T$, we obtain the equation in zero approximation which is identical with the equation for the laminar flow, and the equation in a first approximation

$$F_1''' + 2F_0F_1'' - 6F_0'F_1' + 6F_0''F_1 + 4F_0'F_1 + 2F_0''F_1 - 4F_0' = 0 \qquad (18.73)$$

etc., for the higher approximations.

The results of calculation for the first approximation are also given in Fig. 18.11. In the case of the turbulent jet, in the relative coordinates chosen, the influence of the magnetic parameter which results in a damping of the jet more rapidly than with $H_0 = 0$, proves to be a little weaker than with a laminar jet. This conclusion becomes obvious from Fig. 18.11b where we compared the relative values (scale: value of velocity on the axis of a jet of nonconducting fluid) of the maximum velocity as functions of the parameter $\bar{N}$. From the qualitative point of view, the damping effect of the magnetic field applied is the same in both cases, the laminar and the turbulent jets (with the assumptions adopted on the turbulent viscosity).

The results given are of well-known independent interest. They illustrate clearly not only the specific nature of the problems of magnetohydrodynamics but also the generality of the methods of solving them; these methods are developed in the first parts of the book for the calculation of jets of nonconducting fluid. The further development of the

theory of jets of conducting fluids, as one of the sections dealing with applied gasdynamics, must be connected with experimental investigations of these flows.

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89, 90, 91, 101, 105, 106, 121, 122, 123, 235, 253, 274, 275, 286, 315, 316.

Manuscript Page No.	[Footnotes]
435	It stands to reason that we are here concerned with a torch where the degree of combustion is very high, almost completely.
440	All denotations in the SI system of units.
465	This relationship is conventional. With given values of σ, μ_m, ρ, and v, three quantities (N, A, and $\bar{K}$) are interrelated by two equations. Therefore, when one of them is given, the two others can be determined (within the limits of existence of the solution).

SUPPLEMENT ON THE OCCASION OF PROOFREADING

(To Section 15.5 "Axisymmetric gas jet with parallel flow of Chapter 15 "Calculation and comparison with experiment")

In experimental papers published at the end of 1964 by American researchers data are contained on the expansion of axisymmetric turbulent gas jets in a parallel air flow. The experiments were carried out with jets of various gases, hydrogen, carbon dioxide, etc. Without giving here the results of a detailed comparison of the experimental and theoretical data, we restrict ourselves to some remarks.

In Alpineri's article (*Alpineri L.J.*, Turbulent Mixing of Coaxial Jets, AIAA Journal, Vol. 2, Vol. 2, No. 9, 54-63, 1964) data are given which characterize the turbulent mixing with various values of the ratio of velocities in jet and flow u_∞/u_0 and of the ratio of mass flow densities $\frac{\rho_\infty u_\infty}{\rho_0 u_0}$ (according to our own denotations). The results of comparison prove clearly that the mixing intensity is *minimum* when the parameter $m = \frac{\rho_\infty u_\infty^2}{\rho_0 u_0^2}$ tends to unity, i.e., when the momentum flux densities ρu^2 are the same in jet and flow. In contrast to this, with the same values of the velocity $\frac{u_\infty}{u_0} = 1$ (or with $\frac{\rho_\infty u_\infty}{\rho_0 u_0} = 1$) the intensity of turbulent mixing is very high (and it varies monotonically when these quantities pass through a value equal to unity). By way of example we show in a table (see below) the values of the relative gas concentration on the jet axis in three cross sections, for different values of the parameter m (ordered according to its growth). The data compiled in the table correspond to the right branch of the curves in

$\frac{\rho_{oo}u_{oo}^2}{\rho_0 u_0^2}$	$\Delta c_m/\Delta c_0$			$\frac{u_{oo}}{u_0}$	$\frac{\rho_{oo}u_{oc}}{\rho_0 u_0}$	$\frac{\rho_{oc}}{\rho_0}$	Примечание 1
	$\frac{x}{r_0} \approx 10$	≈ 15	≈ 20				
1,1	0,96	0,75	0,57	1,29	0,86	$\approx 0,65 (CO_2)$	Данные с рис. 10 статьи Альпиньери 2
1,6	0,8	0,6	0,4	1,54	1,05		
3,2	0,55	0,3	0,2	2,12	1,51		
10,7	0,14	0,07	0,04	0,8	13,5	$\approx 16 (H_2)$	
18,4	0,10	0,05	0,03	1,05	17,6		
39	0,06	0,03	0,01	1,51	25		

1) Note; 2) data from Fig. 10 of Alpineri's article.

Fig. 13.1 of this book (beyond the maximum, i.e., with $m > 1$). They indicate the validity of the assumption that the decisive part in the process of nonisotropic turbulent mixing of gases is played by ρu^2. In spite of the considerable spread of data an analogous result was also obtained by Zakkay et al. (*Zakkay V., Krause E., Woo S.D.L.,* Turbulent Transport Properties For Axisymmetric Heterogeneous Mixing, AIAA Journal, Vol.2, No.11, 1928-1937, 1964). It should also be noted that in recent experiments by Sh.A. Yershin, in the range of m-values between 0.2 and 5 for the efflux of hot jets in a parallel flow the results obtained are the same as given in Fig. 13.1, including the efflux conditions with equal values of velocity and mass flux density.

Alpineri's data also permit the conclusion that the intensity of turublent mass transfer (turbulent diffusion) is higher than the intensity of turbulent momentum transfer. In other words, the so-called turbulent diffusion Prandtl number $Pr_{T.dif}$ (or the turbulent Schmidt number S_T), as also Pr_T for energy transfer, are smaller than unity. The approximate value of $\frac{\nu_T}{D_T} \approx \frac{\nu_T}{a_T}$ also lies within the limits of 0.5 and 1.

In the papers mentioned the authors tried to find an empirical

expression for the coefficients of turbulent transfer, remarking that the usual Prandtl formulas (with σ = const) are inapplicable for a compressible gas. At the same time Alpineri (following Ferry, Libby et al.) had recourse to a linearization of the boundary layer equations and their reduction to an equation of the type of the heat-conduction equation in the effective plane of the variables $\xi = \xi(x)$ and (streamfunction). The evaluation shows that the data by Alpineri et al. are in good agreement with the mathematical model developed in the test, i.e., with the method of the equivalent problem of thermal conductivity, in other words, the transformation to a parabolic equation in the effective plane of $\xi = \xi(x)$ and $\eta \approx \psi$ is admissible.

The considerations and conclusions of the Chapters 14 and 15 are thus verified within a wide range of variation of experimental conditions.

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LIST OF MAIN DESIGNATIONS

1. v	is the vector of velocity.
2. x, y, z	are Cartesian coordinates.
x, r, φ	are cylindrical coordinates (Part 1).
x, y, φ	are cylindrical coordinates (Parts 2 and 3).
R, θ, φ	are spherical coordinates ($\omega = \cos\theta$ — Part 1).
3. u, v, w	are the velocity components along the x, r, φ axes (and also along the x, y, z axes in Division 15.4).
4. p, ρ, T, T_0	are the pressure, the density, the temperature, and the stagnation temperature of the fluid or the gas.
5. μ, ν, λ, a, D	are the dynamic and kinematic viscosity coefficients, the heat-conduction coefficient, the temperature-conduction coefficient, and the diffusion coefficient (with "T" subscripts, e.g., ν_T, effective turbulent values).
6. $c_p, R, \kappa = \frac{c_p}{c_v}$	are the specific heat constant pressure, the specific gas constant, the adiabatic exponent.
7. $i = c_pT, i_0 = c_p T_0$	are the enthalpy, and the stagnation enthalpy.
8. μ_m, ν_m, σ	are the magnetic permeability, the magnetic viscosity, and the electric conductivity.
9. M, Re, Pr, Nu	are the Mach number M, the Reynolds number, Prandtl number, and the Nusselt number.
10. $\mathrm{Re}_m, \mathrm{Pr}_m$	are the magnetic Reynolds and Prandtl numbers (Chapter 18).
11. τ, q	are the frictional stress, and the heat flow.
12. J, Q, M, E, G	are the momentum, enthalpy, moment of momentum and kinetic energy flows, and the mass flow rate of the fluid.
13. $B = \mu_m H, H, j$	are the magnetic induction, the magnetic

field strength, and the current density (Chapter 18).

14. α, β, γ, δ, ϵ, $\sigma = \beta/\alpha$ are constants (power exponents) in similarity transformations

15. A, B, C, D, Γ, H, etc. are constants.

16. ξ, η are the Dorodnitsyn variables (Chapter 18) independent variables in the equivalent problem method (Chapter 14,), etc.

17. φ, F, Φ_1, $\mathcal{F}$, g, h, etc. are the reduced coordinate ($\varphi = Byx^{\beta}$) and functions of the coordinate φ.

18. $m_u = \frac{u_2}{u_1}$, $m_i = \frac{T_2}{T_1}$, $\omega = \rho_\infty/\rho$ are parameters (also $m = (\rho u^2)_\infty/\rho u^2$) in §15.5).

19. φ is the stream function.

20. l is the Prandtl mixing length.

21. N, S are the magnetic interaction parameters (Chapter 18).

22. Ω is the stoichiometric reaction coefficient (Chapter 18).

23. U, V, J, τ, r are the reduced variables (Chapter 13).

24. δ_{ik} is the unit tensor.

MAIN SUBSCRIPTS

0	when the jet leaves the nozzle (also for the stagnation parameters).
m	on the jet axis.
∞	in the surrounding medium (in particular, in the comoving flow).
T	means pertaining to turbulence.
w	at the wall surface.
ϕ	at the combustion front (Chapter 17).